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Two-body problem on two-point homogeneous spaces, invariant differential operators and the mass center concept

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Abstract

We consider the two-body problem with central interaction on two-point homogeneous spaces from the point of view of the invariant differential operators theory. The representation of the two-particle Hamiltonian in terms of the radial differential operator and invariant operators on the symmetry group is found. The connection of different mass center definitions for these spaces to the obtained expression for Hamiltonian operator is studied.

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1. Introduction

The purpose of this paper is to provide a comprehensive treatment of the quantum-mechanical two-body problem on Riemannian two-point homogeneous spaces from the point of view of the theory of invariant differential operators developed by Helgason [1] and others (see [2–4] and references therein).

Let M be a Riemannian manifolds with an action of an isometry group G on it. We assume that G-orbits in M of a maximal dimension ℓ are isomorphic to each other, their union is an open dense submanifold M', the set $M \setminus M'$ has zero measure, and $M' = W \times \mathcal{O}$, where $\mathcal{O}$ is a G-orbit of a maximal dimension, and W is a submanifold of M' transversal to all

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G-orbits of the dimension ℓ. This situation is a typical one [5], and we have the isomorphism of measurable sets $(M, \mu) \cong (W, \nu) \times (\mathcal{O}, \mu_G)$, where μ is the G-invariant measure on the manifold M, generated by its metric, ν is some measure on W, and μ_G is a G-invariant measure on $\mathcal{O}$. It implies the isomorphisms (see for example, Theorem II.10 in [6])

$$\mathcal{H} := \mathcal{L}^2(M, \mathrm{d}\mu) = \mathcal{L}^2(W \times \mathcal{O}, \mathrm{d}\nu \otimes \mathrm{d}\mu_G) = \mathcal{L}^2(W, \mathrm{d}\nu) \otimes \mathcal{L}^2(\mathcal{O}, \mathrm{d}\mu_G). \tag{1}$$

Under these assumptions, an invariant differential operator D on M' admits an explicitly symmetric decomposition of the form:

$$D = D_{\mathrm{T}} + \sum_{(i)} D_{(i)} \circ X_{i_1}^+ \circ \cdots \circ X_{i_r}^+ \equiv D_{\mathrm{T}} + \sum_{(i)} D_{(i)} \circ \Box_{(i)}, \tag{2}$$

where X_i^+ is a differential operator of the first-order, corresponding to the action of the one parametric subgroup $\exp(tX_i)$, $X_i \in \mathfrak{g}$, of the group G on the space M'; D_{T} and $D_{(i)}$ are transversal operators with respect to some manifold W which is in transverse position with respect to orbits of the group G in M'; here D_{T} is called the transversal part of D (see Theorem 3.4, Chapter II, [2]). Operators $\Box_{(i)}$ are invariant on orbits of the group G in space M'. Such operators can be naturally expressed in terms of the Lie algebra $\mathfrak{g}$ of the group G. In this paper we assume M and G to be connected. Such an expression of invariant differential operators corresponds to a general approach to invariant differential geometrical objects on homogeneous spaces. These objects have the simplest form in the basis of Killing vector fields [7,8]. For invariant metrics on Lie groups this approach was developed in [9] in the direction to infinite-dimensional groups. Note that the representation (2) depends on a choice of a transversal manifold W. Now let an operator $D = H$ be a Hamiltonian of some quantum mechanical system on a Riemannian manifold M acting in the space $\mathcal{H}$.

The group G naturally acts on the space $\mathcal{L}^2(\mathcal{O}, \mathrm{d}\mu_G)$ by left-shifts, and if $\mathcal{H}'$ is an invariant subspace of the latter space, then the operator H admits a restriction on the space $\mathcal{L}^2(W, \mathrm{d}\nu) \otimes \mathcal{H}'$. For a compact group G we can expand the space $\mathcal{L}^2(\mathcal{O}, \mathrm{d}\mu_G)$ into the direct sum of spaces of irreducible representations of the group G and obtain from (1)

$$\mathcal{H} = \mathcal{L}^2(W, \mathrm{d}\nu) \otimes (\oplus_i \mathcal{H}'_i) = \oplus_i (\mathcal{L}^2(W, \mathrm{d}\nu) \otimes \mathcal{H}'_i).$$

In this case the Hamiltonian H is expanded in the direct sum of operators in spaces $\mathcal{H}_i = \mathcal{L}^2(W, \mathrm{d}\nu) \otimes \mathcal{H}'_i$. Hence the symmetric quantum mechanical system is reduced to the set of subsystems that are not G-symmetric. This method was described in papers [10–12] without mentioning the expansion (2). On the other hand, it seems to be difficult to manage without the expansion (2) while reducing the quantum mechanical system, because representation theory for the group G gives only the formulae for the action of operators $\Box_{(i)}$ in the space $\mathcal{H}'_i$. Without (2), the calculation of the action of H in the space $\mathcal{H}_i$ requires cumbersome computations.

At the same time, the expansion (2) gives some information on the complexity of reduced subsystems even in the absence of the detailed information about irreducible representations of the group G in the space $\mathcal{L}^2(\mathcal{O}, \mathrm{d}\mu_G)$. For example, if all operators $\Box_{(i)}$ in (2) commute, they have only common eigenfunctions, and the spectral problem for the Hamiltonian H reduces to a set of spectral problems for some scalar differential operators on the manifold W.

Now let $H = H_0 + U$ be a Hamiltonian of the system of two particles on a Riemannian space Q. Here $H_0 = -(1/2m_1)\triangle_1 - (1/2m_2)\triangle_2$ is the free two-particle Hamiltonian (everywhere we put $\hbar = 1$), m_1, m_2 are particle masses, $\triangle_i$ $(i = 1, 2)$ is the Laplace–Beltrami operator on the ith factor of the configuration space $M = Q \times Q$ for this system, U is the interaction potential depending only on a distance between particles, and G is the identity component of an isometry group of the space Q. The group G acts naturally on the space $Q \times Q$ as

$$g : (q_1, q_2) \to (gq_1, gq_2), \quad g \in G, \quad (q_1, q_2) \in Q \times Q.$$

The dimension of a manifold $W \subset M$ in this case is one, or greater, since the group G conserves a distance between two points of the space Q. In other words, the codimension of G-orbits in M is 1 or greater. In this paper we consider the case of two-point homogeneous Riemannian spaces Q for which the latter codimension is equal to 1. On the spaces of constant sectional curvature, which are the special case of two-point homogeneous Riemannian spaces, this problem was considered in [13–15].

This paper is organized as follows. In Section 2 we consider the theory of invariant differential operators emphasizing the facts which will be used later for calculating the two-point Hamiltonian. In Section 3 we find the formula for the Laplace–Beltrami operator in a basis of Killing vector fields. This formula for the basis consisting of Killing vector fields and a radial vector field is then generalized in Section 5. The classification of two-point homogeneous Riemannian spaces is given in Section 4. There is also a construction of a special basis for the Lie algebra $\mathfrak{g}$ of the group G. We use this construction and the formula for the Laplace–Beltrami operator from Section 5 to obtain the expression of the type (2) for the two-particle Hamiltonian on two-point compact homogeneous Riemannian spaces in Section 6. Using the formal correspondence between compact and noncompact two-point homogeneous spaces, in Section 7 we transform the latter expression into the form valid for the noncompact case. The Hamiltonian two-particle functions for two-point homogeneous spaces are considered in Section 8. Different mass center concepts on two-point homogeneous spaces are discussed in Section 9. We study the connection of the mass center concepts to the obtained expressions for quantum and classical Hamiltonians.

2. Invariant differential operators on homogeneous spaces

Let M be a Riemannian G-homogeneous space, $\dim M = m$, $\dim G = N$, $x_0 \in M$, $K_{x_0} \subset G$ a stationary subgroup of a point $x_0 \in M$, and $\mathfrak{k}_{x_0} \subset \mathfrak{g} \equiv T_e G$ the corresponding Lie algebras. Choose a subspace $\mathfrak{p}_{x_0} \subset \mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{p}_{x_0} \oplus \mathfrak{k}_{x_0}$ (a direct sum of linear spaces).

The stationary subgroup K_{x_0} is compact, since it is also the subgroup of the group $\mathbf{SO}(m)$. By the group averaging on K_{x_0} we can define a $\mathrm{Ad}_{K_{x_0}}$-invariant scalar product on $\mathfrak{g}$ and choose the subspace $\mathfrak{p}_{x_0}$ orthogonal to $\mathfrak{k}_{x_0}$ with respect to this product [2,16]. In this case we have $\mathrm{Ad}_{K_{x_0}}(\mathfrak{p}_{x_0}) \subset \mathfrak{p}_{x_0}$, i.e. the space M is reductive.

Identify the space M with the factor space of left conjugate classes of the group G with respect to the subgroup K_{x_0}. Let $\pi : G \to G/K_{x_0}$ be the natural projection. Denote by

$$L_q : q_1 \to qq_1, \qquad R_q : q_1 \to q_1 q, \quad q, q_1 \in G$$

the left and the right shifts on the group G, and by

$$\tau_q : x \to qx, \quad q \in G, \ \ x \in M$$

the action of an element $q \in G$ on M. Obviously, $\pi \circ L_q = \tau_q \circ \pi$, $q \in G$ and $\pi \circ R_q = \pi$, $q \in K_{x_0}$. Let the left and the right shifts act on the space $C^\infty(G)$ as

$$\hat{L}_q(f)(q_1) = f(q^{-1}q_1), \qquad \hat{R}_q(f)(q_1) = f(q_1 q^{-1}), \quad f \in C^\infty(G),$$

The left shift acts on the space $C^\infty(M)$ as

$$\hat{\tau}_q(f)(x) = f(q^{-1}x), \quad f \in C^\infty(M).$$

Then $\hat{L}_{q_1q_2} = \hat{L}_{q_1} \circ \hat{L}_{q_2}$, $\hat{R}_{q_1q_2} = \hat{R}_{q_2} \circ \hat{R}_{q_1}$, $\hat{\tau}_{q_1q_2} = \hat{\tau}_{q_1} \circ \hat{\tau}_{q_2}$, $\hat{L}_{q_1} \circ \hat{R}_{q_2} = \hat{R}_{q_2} \circ \hat{L}_{q_1}$, $q_1, q_2 \in G$.

Let $\mathrm{Diff}(G)$ and $\mathrm{Diff}(M)$ be algebras of differential operators with smooth coefficients on G and M, respectively. Define the action of shifts on operators as

$$\tilde{L}_q(\square) = \hat{L}_q \circ \square \circ \hat{L}_{q^{-1}}, \qquad \tilde{R}_q(\square) = \hat{R}_q \circ \square \circ \hat{R}_{q^{-1}}, \quad \square \in \mathrm{Diff}(G),$$
$$\tilde{\tau}_q(\square) = \hat{\tau}_q \circ \square \circ \hat{\tau}_{q^{-1}}, \quad \square \in \mathrm{Diff}(M).$$

Define the following subalgebras:

$$\mathrm{LDiff}(G) := \{\square \in \mathrm{Diff}(G) | \tilde{L}_q(\square) = \square, \ \forall q \in G\},$$

$$\mathrm{LDiff}(M) := \{\square \in \mathrm{Diff}(M) | \tilde{\tau}_q(\square) = \square, \ \forall q \in G\},$$

$$\mathrm{RDiff}(G) := \{\square \in \mathrm{Diff}(G) | \tilde{R}_q(\square) = \square, \ \forall q \in G\},$$

$$\mathrm{LRDiff}(G) := \{\square \in \mathrm{LDiff}(G) | \tilde{R}_q(\square) = \square, \ \forall q \in G\},$$

$$\mathrm{LDiff}_K(G) := \{\square \in \mathrm{LDiff}(G) | \tilde{R}_q(\square) = \square, \ \forall q \in K\},$$

where K is a subgroup of G. For any algebra $\mathcal{A}$ denote $\mathrm{Z}\mathcal{A}$ the center of $\mathcal{A}$. Let $S(V)$ be a symmetric algebra over a finite dimensional complex space V, i.e. a free commutative algebra over the field $\mathbb{C}$, generated by elements of any basis of V. The adjoint action of the group G on $\mathfrak{g}$ can be naturally extended to the action of G on the algebra $S(\mathfrak{g})$ according to the formula:

$$\mathrm{Ad}_q : Y_1 \cdots Y_i \to \mathrm{Ad}_q(Y_1) \cdots \mathrm{Ad}_q(Y_i), \quad Y_1, \ldots, Y_i \in \mathfrak{g}.$$

Denote by $I(\mathfrak{g})$ the set of all Ad-invariants in $S(\mathfrak{g})$.

Let $e_{m+1}, \ldots, e_N$ be a basis in $\mathfrak{k}$, and $e_1, \ldots, e_N$ a basis in $\mathfrak{g}$. There are corresponding moving frames on the group G consisting, respectively, of the following left- and right-invariant vector fields:

$$X_i^{\mathrm{l}}(q) = \mathrm{d}L_q e_i, \qquad X_i^{\mathrm{r}}(q) = \mathrm{d}R_q e_i, \quad i = 1, \ldots, N, \ \ q \in G.$$

There are also the dual moving frames $X^i_{\mathrm{l}}(q)$, $X^i_{\mathrm{r}}(q)$. In general, we shall denote by Y^{l} and Y^{r} the left- and the right-invariant vector fields, corresponding to an element $Y \in \mathfrak{g}$.

We can consider vector fields as differential operators of the first-order and any differential operator on the group G can be expressed as a polynomial in X_i^{l} or in X_i^{r}, $i = 1, \ldots, N$ with nonconstant coefficients. Define a map

$$\lambda : S(\mathfrak{g}) \to \mathrm{Diff}(G)$$

by the formula

$$(\lambda(P)f)(q) = [P(\partial_1, \ldots, \partial_N) f(q \exp(t_1 e_1 + \cdots + t_N e_N))]_{t=0},$$

where $P \in S(\mathfrak{g})$ on the left-hand side is a polynomial in $e_1, \ldots, e_N$ and on the right-hand side the substitution $e_i \to \partial_i := \partial/\partial t_i, i = 1, \ldots, N$ was made. Here $t = (t_1, \ldots, t_N) \in \mathbb{R}^N$, $f \in C^\infty(G)$.

Theorem 1 (Helgason [2]). *The map* λ *is the unique linear bijection* (*generally not a homomorphism*) *of the algebra* $S(\mathfrak{g})$ *onto the algebra* $\mathrm{LDiff}(G)$ *such that*

$$\lambda((Y)^i) = (Y^{\mathrm{l}})^i = \underbrace{Y^{\mathrm{l}} \circ \cdots \circ Y^{\mathrm{l}}}_{i \text{ times}}, \quad Y \in \mathfrak{g}.$$

Remark 1. The map λ transforms the element $Y_1 \cdots Y_p \in S(\mathfrak{g})$ into the operator

$$\frac{1}{p!} \sum_{\sigma \in \mathfrak{S}_p} Y^{\mathrm{l}}_{\sigma(1)} \circ \cdots \circ Y^{\mathrm{l}}_{\sigma(p)} \in \mathrm{LDiff}(G),$$

where $\mathfrak{S}_p$ is the group consisting of all permutations of p elements. The map λ is called symmetrization. With its help the noncommutative algebra $\mathrm{LDiff}(G)$ is described in terms of the free commutative algebra with N generators $e_1, \ldots, e_N$.

Let $\widetilde{\mathrm{Ad}}_q \Box := \tilde{L}_q \circ \tilde{R}_{q^{-1}}(\Box), \Box \in \mathrm{Diff}(G), q \in G$. It is clear, that $\widetilde{\mathrm{Ad}}_q$ is the automorphism of algebras $\mathrm{Diff}(G)$, $\mathrm{LDiff}(G)$, $\mathrm{RDiff}(G)$. Obviously,

$$\widetilde{\mathrm{Ad}}_q \Box = \tilde{R}_{q^{-1}}(\Box), \quad \Box \in \mathrm{LDiff}(G).$$

By direct calculations we have

$$(\mathrm{Ad}_q Y)^{\mathrm{l}} = \widetilde{\mathrm{Ad}}_q Y^{\mathrm{l}} = \tilde{R}_{q^{-1}} Y^{\mathrm{l}}, \quad \forall Y \in \mathfrak{g}. \tag{3}$$

Define the operation

$$\widetilde{\mathrm{ad}}_Y \Box = Y^{\mathrm{l}} \circ \Box - \Box \circ Y^{\mathrm{l}}, \quad \forall Y \in \mathfrak{g}, \ \Box \in \mathrm{Diff}(G).$$

Then

$$(\mathrm{ad}_Y X)^{\mathrm{l}} = Y^{\mathrm{l}} \circ X^{\mathrm{l}} - X^{\mathrm{l}} \circ Y^{\mathrm{l}} = \widetilde{\mathrm{ad}}_Y X^{\mathrm{l}}, \quad X, Y \in \mathfrak{g}. \tag{4}$$

Evidently, the operation $\widetilde{\mathrm{ad}}$ is a differentiation of algebras $\mathrm{Diff}(G)$ and $\mathrm{LDiff}(G)$. By direct calculations we conclude that the operation

$$\exp(\widetilde{\mathrm{ad}}_Y) D := \sum_{i=0}^{\infty} \frac{1}{i!} \widetilde{\mathrm{ad}}_Y^i (D), \quad D \in \mathrm{Diff}(G)$$

is the automorphism of algebras $\mathrm{Diff}(G)$ and $\mathrm{LDiff}(G)$. It is well known that the operations $\exp(\mathrm{ad}_Y)$ and $\mathrm{Ad}_{\exp(Y)}$ coincide on $\mathfrak{g}$, so using (3) and (4) we see that the operations $\widetilde{\mathrm{Ad}}_{\exp Y}$ and $\exp(\widetilde{\mathrm{ad}}_Y)$ coincide also on operators $X^{\mathrm{l}} \in \mathrm{LDiff}(G)$, $X \in \mathfrak{g}$. Since the operators $\widetilde{\mathrm{Ad}}_{\exp Y}$, $\exp(\widetilde{\mathrm{ad}}_Y)$ are automorphisms of the algebra $\mathrm{LDiff}(G)$ and the operators X_i^{l} are the generators of the algebra $\mathrm{LDiff}(G)$ according to Theorem 1, the equality

$$\widetilde{\mathrm{Ad}}_{\exp Y} = \exp(\widetilde{\mathrm{ad}}_Y),$$

holds everywhere in $\mathrm{LDiff}(G)$.

Functions on space M are in one to one correspondence with the functions on group G that are invariant under the right action of the subgroup K_{x_0}. This correspondence is defined by the formula $\zeta : f \to \tilde{f} := f \circ \pi$, where f is a function on space M and $\tilde{f}$ the corresponding function on group G. If f is smooth, then so is $\tilde{f}$. Define a map

$$\eta : \mathrm{LDiff}_{K_{x_0}}(G) \to \mathrm{LDiff}(M)$$

by the formula

$$\eta(\Box) f = \zeta^{-1} \circ \Box \circ \zeta(f), \quad f \in C^\infty(M), \; \Box \in \mathrm{LDiff}_{K_{x_0}}(G).$$

This map is well defined, since the function $\Box \circ \zeta(f)$ is right-invariant with respect to the subgroup K_{x_0}. Evidently, the map η is a homomorphism.

Suppose now that $[\mathfrak{p}_{x_0}, \mathfrak{k}_{x_0}] \subset \mathfrak{p}_{x_0}$, so $\mathrm{Ad}_{K_{x_0}} \mathfrak{p}_{x_0} \subset \mathfrak{p}_{x_0}$. In some neighborhood of the point $x_0 \in M$ we can define coordinates $\{x^1, \dots, x^m\}$, which correspond to the point $\pi\left(\exp\left(\sum_{i=1}^m x^i e_i\right)\right)$. The expression of an operator $\Box \in \mathrm{LDiff}(M)$ at the point x_0 is a polynomial $P((\partial/\partial x^1), \dots, (\partial/\partial x^m))$. Define a map:

$$\varkappa : \mathrm{LDiff}(M) \to S(\mathfrak{p}_{x_0}),$$

by the formula $\varkappa(\Box) = P(e_1, \dots, e_m) \in S(\mathfrak{p}_{x_0})$. For any $f \in C^\infty(M)$ and $\forall q \in G$ we have

$$\begin{aligned}(\Box f)(q x_0) &= \tau_{q^{-1}} \circ \Box(f)(x_0) = \Box \circ \tau_{q^{-1}}(f)(x_0)\\ &= \left[P\left(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^m}\right) f\left(\pi\left(q \exp\left(\sum_{i=1}^m x^i e_i\right)\right)\right)\right]_{x^i=0}\\ &= \left[P\left(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^m}\right) \tilde{f}\left(q \exp\left(\sum_{i=1}^m x^i e_i\right)\right)\right]_{x^i=0}.\end{aligned}$$

In particular, for $q \in K_{x_0}$ it holds

$$\begin{aligned}(\Box f)(x_0) &= \left[P\left(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^m}\right) \tilde{f}\left(\exp\left(\sum_{i=1}^m x^i \,\mathrm{Ad}_q e_i\right)\right)\right]_{x^i=0}\\ &= \left[\tilde{P}\left(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^m}\right) \tilde{f}\left(\exp\left(\sum_{i=1}^m x^i e_i\right)\right)\right]_{x^i=0},\end{aligned}$$

where $\tilde{P}(e_1, \dots, e_m) = P(\mathrm{Ad}_q e_1, \dots, \mathrm{Ad}_q e_m) = \mathrm{Ad}_q P(e_1, \dots, e_m)$, since the map $\varkappa$ does not depend on a choice of the basis for the space $\mathfrak{p}_{x_0}$ and, in particular, it is not changed by the transition to the basis $\mathrm{Ad}_q e_i$. On the other hand, polynomials P and $\tilde{P}$ are two expressions of the operator $\Box$ at the point x_0, so $\tilde{P}(e_1, \dots, e_m) = P(e_1, \dots, e_m)$, i.e. $P(e_1, \dots, e_m) \in I(\mathfrak{p}_{x_0})$, where $I(\mathfrak{p}_{x_0}) \subset S(\mathfrak{p}_{x_0})$ is the set of $\mathrm{Ad}_{K_{x_0}}$ invariants. Note that $I(\mathfrak{p}_{x_0}) \subset I(\mathfrak{g})$. Hence we have the following commutative diagram:

$$\begin{tikzcd} I(\mathfrak{p}_{x_0}) \arrow[r, "\lambda"] & \mathrm{LDiff}_{K_{x_0}}(G) \arrow[d, "\eta"] \\ \mathrm{LDiff}(M) \arrow[u, "\varkappa"] & \mathrm{LDiff}(M) \arrow[l, "\mathrm{id}"'] \end{tikzcd}$$

The structure of the algebra $\mathrm{LDiff}(M)$ was studied in [1,2] with the help of maps λ and η. We are interested in the representation of a fixed operator from the algebra $\mathrm{LDiff}(M)$ by a polynomial from the set $I(\mathfrak{p}_{x_0})$. We have constructed the map $\varkappa$ in order to find such a representation. From the definition of the map λ we see that $\eta\circ\lambda\circ\varkappa = \mathrm{id}$ and $\varkappa\circ\eta\circ\lambda = \mathrm{id}$. Hence the maps $\varkappa$, λ are bijective, the map η is surjective, and the following expansion holds:

$$\mathrm{LDiff}_{K_{x_0}}(G) = \lambda(I(\mathfrak{p}_{x_0})) \oplus \ker \eta.$$

Denote by $\mathrm{LDiff}_{\mathfrak{k}}(G)$ the left ideal in the algebra $\mathrm{LDiff}(G)$, generated by operators X_i^{l}, $i = m+1, \dots, N$ and let

$$\mathrm{LDiff}_{K_{x_0}}^{\mathfrak{k}}(G) := \mathrm{LDiff}_{K_{x_0}}(G) \cap \mathrm{LDiff}_{\mathfrak{k}}(G).$$

Lemma 1 (Helgason [2]). *The algebra* $\mathrm{LDiff}(G)$ *admits the following expansion*

$$\mathrm{LDiff}(G) = \mathrm{LDiff}_{\mathfrak{k}}(G) \oplus \lambda(S(\mathfrak{p}_{x_0})).$$

Theorem 2. *If* $[\mathfrak{p}_{x_0}, \mathfrak{k}_{x_0}] \subset \mathfrak{p}_{x_0}$, *then* $\ker \eta = \mathrm{LDiff}_{K_{x_0}}^{\mathfrak{k}}(G)$.

Remark 2. We shall use an operator $\lambda \circ \varkappa(\Box) \in \mathrm{LDiff}_{K_{x_0}}(G)$ as a lift $\tilde{\Box}$ of an operator $\Box \in \mathrm{LDiff}(M)$ onto the group G.

From the construction of the maps λ, $\varkappa$ and Remark 1 we obtain that if $\Box|_{x_0} = P(X_1, \dots, X_m)|_{x_0}$, where P is a polynomial invariant with respect to any permutation of its arguments, then $\lambda \circ \varkappa(\Box) = P(X_1^{\mathrm{l}}, \dots, X_m^{\mathrm{l}})$. This formula for the lift depends on the expansion $\mathfrak{g} = \mathfrak{k}_{x_0} \oplus \mathfrak{p}_{x_0}$.

There exists a unique (up to a constant factor) left- (or right-) invariant measure on any Lie group (the Haar measure [2,17]). Denote by μ_G some left-invariant Haar measure on G. A measure on the space M, generated by a G-invariant metric is also G-invariant. All G-invariant measures on M are proportional [2], and we define such a measure if we put $\mu_M(V) = \mu_G(\pi^{-1}(V))$ for any compact set $V \in M$. The group K_{x_0} is compact, so the set $\pi^{-1}(V)$ is also compact, and $\mu_G(\pi^{-1}(V)) < \infty$. The measure μ_M is left-invariant, since the measure μ_G is left-invariant.

On all unimodular groups left-invariant measures are also right-invariant. The change of the point $x_0 \in M$ to $x_1 = qx_0$, $q \in G$ leads to the change of the pullback $\pi^{-1}(V)$ to

$\pi^{-1}(V)q^{-1}$, while identifying M with G/K_{x_0}. Therefore the G-invariant measure μ_M for the unimodular group G does not depend on the choice of x_0.

3. Laplace–Beltrami operator in a moving frame

Now we shall find the polynomial P mentioned in Remark 2 above, corresponding to the Laplace–Beltrami operator on the space M. First, let us obtain the expression for the Laplace–Beltrami operator in arbitrary moving frame.

Here we do not regard M as a homogeneous manifold with respect to the isometry group until the homogeneity is declared explicitly. Denote the metric on M by g. Let x^i, $i = 1, \ldots, m$ be the local coordinates in a domain $U \subset M$ and $g_{ij}\, dx^i\, dx^j$ be the expression of the metric g on U. The Laplace–Beltrami operator generated by the metric g has the following form on U:

$$\Delta_g = (\gamma)^{-1/2} \frac{\partial}{\partial x^i} \left(\sqrt{\gamma} g^{ij} \frac{\partial}{\partial x^j} \right), \tag{5}$$

where $\gamma = |\det g_{ij}|$, and $g^{ij}(x)$ is the inverse of the matrix $g_{ij}(x)$. The operator Δ is conserved by all isometries of the space M. Conversely, if the operator Δ is conserved by some diffeomorphism of the space M, then this diffeomorphism is the isometry [2].

Let $\xi_i = \phi_i^k(x)(\partial/\partial x^k)$, $i = 1, \ldots, m$ be vector fields on U forming a moving frame. Any vector field is a differential operator of the first-order. Using the operation of composition and nonconstant coefficients we can express any differential operator on U via this moving frame. The range for all indices in this section is $1, \ldots, m$.

Let ψ_j^i be the inverse of the matrix ϕ_i^k. Then $\partial/\partial x^k = \psi_k^m \xi_m$, and $\hat{g}_{i,j} := g(\xi_i, \xi_j) = \phi_i^k \phi_j^m g_{km}$ are the coefficients of the metric g with respect to the moving frame ξ_i. This implies that $\hat{g}^{ij} = \psi_k^i g^{kn} \psi_n^j$ and $\hat{\gamma} := |\det \hat{g}_{ij}| = \phi^2 \gamma$, where $\phi = \det \phi_i^k$. Substituting these formulae in (5), we obtain:

$$\begin{aligned}
\Delta_g &= (\hat{\gamma})^{-1/2} \phi \psi_i^q \xi_q \circ (\phi^{-1} \hat{\gamma}^{1/2} \phi_k^i \hat{g}^{kn} \phi_n^j \psi_j^p \xi_p) \\
&= (\hat{\gamma})^{-1/2} \phi \psi_i^q \xi_q \circ (\phi^{-1} \hat{\gamma}^{1/2} \phi_k^i \hat{g}^{kn} \xi_n) \\
&= (\hat{\gamma})^{-1/2} \psi_i^q \phi_k^i \xi_q \circ (\hat{\gamma}^{1/2} \hat{g}^{kn} \xi_n) + \phi \psi_i^q \hat{g}^{kn} \xi_q (\phi^{-1} \phi_k^i) \xi_n \\
&= (\hat{\gamma})^{-1/2} \xi_k \circ (\hat{\gamma}^{1/2} \hat{g}^{kn} \xi_n) + \hat{g}^{kn} L_k \xi_n,
\end{aligned}$$

where

$$L_k = \phi \psi_i^q \xi_q (\phi^{-1} \phi_k^i) = \psi_i^q \xi_q (\phi_k^i) + \phi \xi_k (\phi^{-1}) = \psi_i^q \xi_q (\phi_k^i) - \phi^{-1} \xi_k (\phi).$$

Denote $\Phi(x) = \|\phi_k^i(x)\|$. Then using the well known formula $\det \exp(A) = \exp(\operatorname{Tr} A)$, where A is an arbitrary complex matrix, we get

$$\begin{aligned}
\phi^{-1} \xi_k(\phi) &= \xi_k(\ln \phi) = \xi_k(\ln \exp(\operatorname{Tr} \ln \Phi)) = \xi_k(\operatorname{Tr} \ln \Phi) \\
&= \operatorname{Tr}(\Phi^{-1} \xi_k(\Phi)) = \psi_i^q \xi_k(\phi_q^i),
\end{aligned} \tag{6}$$

since $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$ for any square matrices A and B. On the other hand, the following equations for commutators of vector fields

$$[\xi_i, \xi_j] = \xi_i(\phi_j^k)\frac{\partial}{\partial x^k} - \xi_j(\phi_i^k)\frac{\partial}{\partial x^k} = (\xi_i(\phi_j^k) - \xi_j(\phi_i^k))\psi_k^q\xi_q =: c_{ij}^q\xi_q$$

define the functions

$$c_{ij}^q = (\xi_i(\phi_j^k) - \xi_j(\phi_i^k))\psi_k^q$$

on U. So, in view of (6), we obtain

$$L_k = (\xi_q(\phi_k^i) - \xi_k(\phi_q^i))\psi_i^q = c_{qk}^q.$$

Thus we get the formula for the Laplace–Beltrami operator in the moving frame ξ_i

$$\Delta_g = (\hat{\gamma})^{-1/2}\xi_q \circ (\sqrt{\hat{\gamma}}\hat{g}^{qn}\xi_n) + \hat{g}^{kn}c_{qk}^q\xi_n. \tag{7}$$

Let now ξ_i be Killing vector fields for the metric g in U. Transform the formula (7) to the form $\Delta_g = a^{ij}\xi_i \circ \xi_j + b^i\xi_i$. It is clear that $a^{ij} = \hat{g}^{ij}$ and we only have to find the coefficients b^i. The well known formula

$$X(g(Y, Z)) = (\pounds_X g)(Y, Z) + g([X, Y], Z) + g(Y, [X, Z]),$$

where X, Y, Z are vector fields on M, $\pounds_X$ is a Lie derivative with respect to a field X, and formulae $\pounds_{\xi_k} g = 0$, (6), (7) imply

$$\begin{aligned}
b^i\hat{g}_{ij} &= \hat{\gamma}^{-1/2}\xi_k(\hat{\gamma}^{1/2}\hat{g}^{ki})\hat{g}_{ij} + \hat{g}^{ki}c_{qk}^q\hat{g}_{ij}\\
&= \xi_k(\hat{g}^{ki}\hat{g}_{ij}) - \xi_k(\hat{g}_{ij})\hat{g}^{ki} + \frac{1}{2\hat{\gamma}}\xi_k(\hat{\gamma})\hat{g}^{ki}\hat{g}_{ij} + c_{qk}^q\delta_j^k\\
&= \xi_k(\delta_j^k) - \xi_k(g(\xi_i, \xi_j))\hat{g}^{ki} + \frac{1}{2}\hat{g}^{qi}\xi_k(\hat{g}_{qi})\delta_j^k + c_{qj}^q\\
&= -g([\xi_k, \xi_i], \xi_j)\hat{g}^{ki} - g(\xi_i, [\xi_k, \xi_j])\hat{g}^{ki}\\
&\quad + \frac{1}{2}\hat{g}^{qi}g([\xi_j, \xi_q], \xi_i) + \frac{1}{2}\hat{g}^{qi}g(\xi_q, [\xi_j, \xi_i]) + c_{qj}^q\\
&= -\hat{g}_{iq}c_{kj}^q\hat{g}^{ki} + \frac{1}{2}\hat{g}^{qi}c_{jq}^k\hat{g}_{ki} + \frac{1}{2}\hat{g}^{qi}c_{ji}^k\hat{g}_{qk} + c_{qj}^q\\
&= -c_{qj}^q + \frac{1}{2}c_{jq}^q + \frac{1}{2}c_{jq}^q + c_{qj}^q = c_{jq}^q,
\end{aligned} \tag{8}$$

taking into account the antisymmetry of the tensor c_{ki}^q with respect to lower indices. Thus we obtain $b^i = c_{jq}^q\hat{g}^{ji}$. We can summarize this reasoning in the following proposition:

Proposition 1. *In the moving frame ξ_i the Laplace–Beltrami operator have the following form*

$$\Delta_g = \hat{g}^{ij}\xi_i \circ \xi_j + c_{jq}^q\hat{g}^{ji}\xi_i.$$

If the space M is homogeneous and $\xi_i = X_i$ *in notations of* Section 2, *then* Remark 2 *implies that the lift of the operator* Δ_g *onto the group G has the form*:

$$\tilde{\Delta}_g = \hat{g}^{ij}|_{x_0} X_i^{\mathrm{l}} \circ X_j^{\mathrm{l}} + c_{jq}^q \hat{g}^{ji}|_{x_0} X_i^{\mathrm{l}}.$$

Remark 3. Sometimes vector fields ξ_i can be chosen in such a way that $c_{jq}^q = 0$. In this case we have $\Delta_g = \hat{g}^{ij}\xi_i \circ \xi_j$ and $\tilde{\Delta}_g = \hat{g}^{ij}|_{x_0} X_i^{\mathrm{l}} \circ X_j^{\mathrm{l}}$.

In the sequel, we shall use the expression for the Riemannian connection ∇ in the moving frame ξ_i given by the following lemma:

Lemma 2 (Besse [7], 7.27).

$$g(\nabla_{\xi_i}\xi_j, \xi_k) = \tfrac{1}{2}g(\xi_i, [\xi_j, \xi_k]) + \tfrac{1}{2}g(\xi_j, [\xi_i, \xi_k]) + \tfrac{1}{2}g([\xi_i, \xi_j], \xi_k).$$

In particular, for $i = j$

$$g(\nabla_{\xi_i}\xi_i, \xi_k) = g(\xi_i, [\xi_i, \xi_k]). \tag{9}$$

without summation over index i.

4. Two-point homogeneous Riemannian spaces

The main characteristic for the system of two classical particles is the distance between them. If the configuration space Q is homogeneous and isotropic, this distance is the only geometric invariant for Q. These spaces are called *two-point homogeneous spaces* [18], i.e. any pair of points on such space can be transformed by means of appropriate isometry to any other pair of points with the same distance between them. In the following, Q denotes the two-point homogeneous connected Riemannian space. The classification of these spaces can been found in [19,20] (see also [18,21]) and is as follows:

1. the Euclidean space $\mathbf{E}^n$,
2. the sphere $\mathbf{S}^n$,
3. the real projective space $\mathbf{P}^n(\mathbb{R})$,
4. the complex projective space $\mathbf{P}^n(\mathbb{C})$,
5. the quaternion projective space $\mathbf{P}^n(\mathbb{H})$,
6. the Cayley projective plane $\mathbf{P}^2(\mathrm{Ca})$,
7. the real hyperbolic space (Lobachevski space) $\mathbf{H}^n(\mathbb{R})$,
8. the complex hyperbolic space $\mathbf{H}^n(\mathbb{C})$,
9. the quaternion hyperbolic space $\mathbf{H}^n(\mathbb{H})$,
10. the Cayley hyperbolic plane $\mathbf{H}^2(\mathrm{Ca})$.

There are different equivalent approaches to classification of these spaces. Recall that the *rank of a symmetric space* is the dimension of its maximal flat completely geodesic submanifold.

Theorem 3. *Let Q be a connected Riemannian space, G is the identity component of the isometry group for M, and I_x is a stationary subgroup for a point x. Then the following conditions* 1–3 *are equivalent*

1. *Q is two-point homogeneous*;
2. *the action of the stationary subgroup I_x on all unit spheres T_xQ, $\forall x \in Q$ in the tangent spaces is transitive*; *in other words, Q is isotropic*;
3. *Q is the symmetric space of the rank one*.

If any of these condition is satisfied, then all geodesics on the compact space Q are closed and have the same length. This follows from the homogeneity and isotropy of Q.

This result has been proved in [18, Lemma 8.12.1], [19,20], see also references in [22, p. 535].

Let now Q be the two-point homogeneous compact Riemannian space (i.e. the space of one of the types 2–6). We assume that the point x_0 is fixed (the index x_0 will sometimes be omitted in the following). All geodesics on Q are closed and have the same length equal $2\,\mathrm{diam}\, Q$, where $\mathrm{diam}\, Q$ is the maximal distance between two points of the space Q. Put $R = 2\,\mathrm{diam}\, Q/\pi$ for the space $\mathbf{P}^n(\mathbb{R})$ and $R = \mathrm{diam}\, Q/\pi$ for the other compact two-point homogeneous Riemannian spaces. The maximal sectional curvature of all these spaces is R^{-2} and the minimal sectional curvature of the spaces $\mathbf{P}^n(\mathbb{C})$, $\mathbf{P}^n(\mathbb{H})$, $\mathbf{P}^2(\mathrm{Ca})$ is $(2R)^{-2}$.

Let $\tilde{\gamma}(s)$, $s \in [0, \mathrm{diam}\, Q)$ be some geodesic, which is parameterized by a natural parameter s and $\tilde{\gamma}(0) = x_0$. Since the space Q is symmetric, in the algebra $\mathfrak{g}$ there exists a complementary subspace $\mathfrak{p}$ with respect to the subalgebra $\mathfrak{k}$ such that $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$, $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$. The space $\mathfrak{p}$ can be naturally identified with the space $T_{x_0}Q$. Under this identification the restriction of the Killing form for the algebra $\mathfrak{g}$ onto the space $\mathfrak{p}$ and the scalar product on $T_{x_0}Q$ are proportional. In particular, the decomposition $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ is uniquely determined by the point x_0. Let $(\cdot, \cdot)$ be the scalar product on the algebra $\mathfrak{g}$ such that it is proportional to the Killing form and its restriction onto the subspace $\mathfrak{p} \cong T_{x_0}Q$ coincides with the Riemannian metric g on $T_{x_0}Q$. The inclusions

$$[\mathfrak{p}, [\mathfrak{k}, \mathfrak{p}]] \subset \mathfrak{k}, \qquad [\mathfrak{p}, [\mathfrak{k}, \mathfrak{k}]] \subset \mathfrak{p}$$

and the definition of the Killing form imply that the spaces $\mathfrak{p}$ and $\mathfrak{k}$ are orthogonal to each other with respect to the scalar product $(\cdot, \cdot)$. From the results of [22,23] we can extract the following proposition.

Proposition 2. *The algebra $\mathfrak{g}$ admits the following expansion into the direct sum of subspaces*:

$$\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{k}_0 \oplus \mathfrak{k}_\lambda \oplus \mathfrak{k}_{2\lambda} \oplus \mathfrak{p}_\lambda \oplus \mathfrak{p}_{2\lambda},$$

where $\dim \mathfrak{a} = 1$, λ *is a nontrivial linear form on the space* $\mathfrak{a}$, $\dim \mathfrak{k}_\lambda = \dim \mathfrak{p}_\lambda = q_1$, $\dim \mathfrak{k}_{2\lambda} = \dim \mathfrak{p}_{2\lambda} = q_2$, $\mathfrak{p} = \mathfrak{a} \oplus \mathfrak{p}_\lambda \oplus \mathfrak{p}_{2\lambda}$, $\mathfrak{k} = \mathfrak{k}_0 \oplus \mathfrak{k}_\lambda \oplus \mathfrak{k}_{2\lambda}$; *where $q_1, q_2 \in \{0\} \cup \mathbb{N}$, the subalgebra $\mathfrak{a}$ is the maximal commutative subalgebra in the subspace $\mathfrak{p}$ and it corresponds to the tangent vectors to the geodesic $\tilde{\gamma}$ at the point x_0. Besides, the following inclusions* *are valid*:

$$
\begin{aligned}
&[\mathfrak{a}, \mathfrak{p}_\lambda] \subset \mathfrak{k}_\lambda, \quad [\mathfrak{a}, \mathfrak{f}_\lambda] \subset \mathfrak{p}_\lambda, \quad [\mathfrak{a}, \mathfrak{p}_{2\lambda}] \subset \mathfrak{k}_{2\lambda}, \quad [\mathfrak{a}, \mathfrak{f}_{2\lambda}] \subset \mathfrak{p}_{2\lambda}, \quad [\mathfrak{a}, \mathfrak{k}_0] = 0,\\
&[\mathfrak{k}_\lambda, \mathfrak{p}_\lambda] \subset \mathfrak{p}_{2\lambda} \oplus \mathfrak{a}, \quad [\mathfrak{k}_\lambda, \mathfrak{k}_\lambda] \subset \mathfrak{k}_{2\lambda} \oplus \mathfrak{k}_0, \quad [\mathfrak{p}_\lambda, \mathfrak{p}_\lambda] \subset \mathfrak{k}_{2\lambda} \oplus \mathfrak{k}_0, \quad [\mathfrak{k}_{2\lambda}, \mathfrak{k}_{2\lambda}] \subset \mathfrak{k}_0,\\
&[\mathfrak{p}_{2\lambda}, \mathfrak{p}_{2\lambda}] \subset \mathfrak{k}_0, \quad [\mathfrak{k}_{2\lambda}, \mathfrak{p}_{2\lambda}] \subset \mathfrak{a}, \quad [\mathfrak{k}_\lambda, \mathfrak{k}_{2\lambda}] \subset \mathfrak{k}_\lambda, \quad [\mathfrak{k}_\lambda, \mathfrak{p}_{2\lambda}] \subset \mathfrak{p}_\lambda, \quad [\mathfrak{p}_\lambda, \mathfrak{k}_{2\lambda}] \subset \mathfrak{p}_\lambda,\\
&[\mathfrak{p}_\lambda, \mathfrak{p}_{2\lambda}] \subset \mathfrak{k}_\lambda.
\end{aligned} \tag{10}
$$

Moreover, for any basis $e_{\lambda,i}, i = 1, \dots, q_1$ *in the space* $\mathfrak{p}_\lambda$ *and any basis* $e_{2\lambda,i}, i = 1, \dots, q_2$ *in the space* $\mathfrak{p}_{2\lambda}$ *there are the basis* $f_{\lambda,i}, i = 1, \dots, q_1$ *in the space* $\mathfrak{k}_\lambda$ *and the basis* $f_{2\lambda,i}$, $i = 1, \dots, q_2$ *in the space* $\mathfrak{k}_{2\lambda}$ *such that*

$$
\begin{aligned}
&[Z, e_{\lambda,i}] = -\lambda(Z) f_{\lambda,i}, \qquad [Z, f_{\lambda,i}] = \lambda(Z) e_{\lambda,i}, \quad i = 1, \dots, q_1,\\
&[Z, e_{2\lambda,i}] = -2\lambda(Z) f_{2\lambda,i}, \qquad [Z, f_{2\lambda,i}] = 2\lambda(Z) e_{2\lambda,i}, \quad i = 1, \dots, q_2, \quad \forall Z \in \mathfrak{a}.
\end{aligned} \tag{11}
$$

If a vector $\Lambda \in \mathfrak{a}$ *satisfies the condition* $(\Lambda, \Lambda) = R^2$, *then* $|\lambda(\Lambda)| = 1/2$.

Nonnegative integers q_1 and q_2 are said to be *multiplicities of the space* Q. They characterize Q uniquely up to the exchange $\mathbf{S}^n \leftrightarrow \mathbf{P}^n(\mathbb{R})$. For the spaces $\mathbf{S}^n$ and $\mathbf{P}^n(\mathbb{R})$ we have $q_1 = 0, q_2 = n - 1$; for the space $\mathbf{P}^n(\mathbb{C}) : q_1 = 2n - 2, q_2 = 1$; for the space $\mathbf{P}^n(\mathbb{H}) : q_1 = 4n - 4, q_2 = 3$; and for the space $\mathbf{P}^2(\mathrm{Ca}) : q_1 = 8, q_2 = 7$. Conversely, for the spaces $\mathbf{S}^n$ and $\mathbf{P}^n(\mathbb{R})$ we could reckon that $q_1 = n - 1, q_2 = 0$. Our choice corresponds to the isometries $\mathbf{P}^1(\mathbb{C}) \cong \mathbf{S}^2$, $\mathbf{P}^1(\mathbb{H}) \cong \mathbf{S}^4$.

Remark 4. The space $\mathfrak{a} \oplus \mathfrak{p}_{2\lambda}$ generates in the space Q a completely geodesic submanifold of the constant sectional curvature R^{-2} and dimension $q_2 + 1$. For all the above mentioned spaces Q with the exception of $\mathbf{P}^n(\mathbb{R})$ this submanifold is a sphere. For the space $\mathbf{P}^n(\mathbb{R})$ this submanifold is the space $\mathbf{P}^{q_2+1}(\mathbb{R})$. If $q_1 \neq 0$, the element Λ and an arbitrary nonzero element from the space $\mathfrak{p}_\lambda$ generate in Q a completely geodesic two-dimensional submanifolds of the constant curvature R^{-2}.

Choose a vector $\Lambda \in \mathfrak{a}$ such that $\lambda(\Lambda) = 1/2$. The following proposition easily follows from Proposition 2:

Proposition 3. *The spaces* $\mathfrak{a} \oplus \mathfrak{k}_0$, $\mathfrak{k}_\lambda \oplus \mathfrak{p}_\lambda$, $\mathfrak{k}_{2\lambda} \oplus \mathfrak{p}_{2\lambda}$ *are pairwise orthogonal. One has*

$$
\begin{aligned}
&(e_{\lambda,i}, e_{\lambda,j}) = (f_{\lambda,i}, f_{\lambda,j}), \qquad (e_{\lambda,i}, f_{\lambda,j}) = -(f_{\lambda,i}, e_{\lambda,j}), \quad i, j = 1, \dots, q_1,\\
&(e_{2\lambda,i}, e_{2\lambda,j}) = (f_{2\lambda,i}, f_{2\lambda,j}), \qquad (e_{2\lambda,i}, f_{2\lambda,j}) = -(f_{2\lambda,i}, e_{2\lambda,j}),\\
&\qquad i, j = 1, \dots, q_2.
\end{aligned} \tag{12}
$$

In particular,

$$
(e_{\lambda,i}, f_{\lambda,i}) = 0, \quad i = 1, \dots, q_1, \qquad (e_{2\lambda,j}, f_{2\lambda,j}) = 0, \quad j = 1, \dots, q_2.
$$

Proof. The Jacobi identity and the Ad_G-invariance of the metric $(\cdot,\cdot)$ imply that the operator $T_\Lambda : X \to [\Lambda, [\Lambda, X]]$ is symmetric on the space $\mathfrak{g}$. This operator has the following eigenspaces $\mathfrak{a} \oplus \mathfrak{k}_0$, $\mathfrak{k}_\lambda \oplus \mathfrak{p}_\lambda$, $\mathfrak{k}_{2\lambda} \oplus \mathfrak{p}_{2\lambda}$ with eigenvalues $0, -\lambda^2(\Lambda) = -1/4, -4\lambda^2(\Lambda) = -1$, respectively. Thus, these eigenspaces are orthogonal to each other. Let us prove the first equality from (12). The Ad_G-invariance of the metric $(\cdot,\cdot)$ and the equality (11) give

$$\lambda(\Lambda)(e_{\lambda,i}, e_{\lambda,j}) = ([\Lambda, f_{\lambda,i}], e_{\lambda,j}) = -(f_{\lambda,i}, [\Lambda, e_{\lambda,j}]) = \lambda(\Lambda)(f_{\lambda,i}, f_{\lambda,j}).$$

Similar calculations prove other equalities from (12). □

The Jacobi identity and formulae (11) give $[Z, [e_{\lambda,i}, f_{\lambda,i}]] = 0$. Thus the relations (10) give $[e_{\lambda,i}, f_{\lambda,i}] \in \mathfrak{a}$. Let $[e_{\lambda,i}, f_{\lambda,i}] =: \varkappa_i \Lambda$. The Ad_G-invariance of the metric $(\cdot,\cdot)$ leads to

$$0 = (\Lambda, [e_{\lambda,i}, f_{\lambda,i}]) + ([e_{\lambda,i}, \Lambda], f_{\lambda,i}) = \varkappa_i(\Lambda, \Lambda) + \lambda(\Lambda)(f_{\lambda,i}, f_{\lambda,i})$$

and

$$\varkappa_i = -\frac{\lambda(\Lambda)}{(\Lambda, \Lambda)}(f_{\lambda,i}, f_{\lambda,i}) = -\frac{\lambda(\Lambda)}{(\Lambda, \Lambda)}(e_{\lambda,i}, e_{\lambda,i}).$$

Similarly, we get

$$[e_{2\lambda,i}, f_{2\lambda,i}] = -\frac{2\lambda(\Lambda)}{(\Lambda, \Lambda)}(f_{2\lambda,i}, f_{2\lambda,i})\Lambda = -\frac{2\lambda(\Lambda)}{(\Lambda, \Lambda)}(e_{2\lambda,i}, e_{2\lambda,i})\Lambda.$$

Using the freedom provided by Proposition 2, we choose the bases $\{e_{\lambda,i}\}_{i=1}^{q_1}$ in the space $\mathfrak{p}_\lambda$ and $\{e_{2\lambda,j}\}_{j=1}^{q_2}$ in the space $\mathfrak{p}_{2\lambda}$ to be orthogonal with norms of all their elements equal R. Thus, the elements $\Lambda, e_{\lambda,i}, e_{2\lambda,j}, i = 1, \dots, q_1, j = 1, \dots, q_2$ form the orthogonal basis in the space $\mathfrak{p}$ and the elements $f_{\lambda,i}, f_{2\lambda,j}, i = 1, \dots, q_1, j = 1, \dots, q_2$ form the orthogonal basis in the space $\mathfrak{k}_\lambda \oplus \mathfrak{k}_{2\lambda}$, due to Proposition 3. Note that $(f_{\lambda,i}, f_{\lambda,i}) = R^2, i = 1, \dots, q_1$, $(f_{2\lambda,j}, f_{2\lambda,j}) = R^2, j = 1, \dots, q_2$.

Proposition 4.

1. *The relations* (11) *can be rewritten in the following form*:

$$\begin{aligned} &[\Lambda, e_{\lambda,i}] = -\tfrac{1}{2} f_{\lambda,i}, \qquad [\Lambda, f_{\lambda,i}] = \tfrac{1}{2} e_{\lambda,i}, \qquad [e_{\lambda,i}, f_{\lambda,i}] = -\tfrac{1}{2}\Lambda,\\ &(e_{\lambda,i}, e_{\lambda,j}) = (f_{\lambda,i}, f_{\lambda,j}) = \delta_{ij} R^2, \quad i, j = 1, \dots, q_1,\\ &[\Lambda, e_{2\lambda,i}] = -f_{2\lambda,i}, \qquad [\Lambda, f_{2\lambda,i}] = e_{2\lambda,i}, \qquad [e_{2\lambda,i}, f_{2\lambda,i}] = -\Lambda,\\ &(e_{2\lambda,i}, e_{2\lambda,j}) = (f_{2\lambda,i}, f_{2\lambda,j}) = \delta_{ij} R^2, \quad i, j = 1, \dots, q_2, \quad (\Lambda, \Lambda) = R^2. \end{aligned} \tag{13}$$

2. *Let X and Y be some elements from the basis*

$$\Lambda, e_{\lambda,i}, f_{\lambda,i}, e_{2\lambda,j}, f_{2\lambda,j}, \quad i = 1, \dots, q_1, \quad j = 1, \dots, q_2 \tag{14}$$

of the space $\mathfrak{m} := \mathfrak{a} \oplus \mathfrak{k}_\lambda \oplus \mathfrak{k}_{2\lambda} \oplus \mathfrak{p}_\lambda \oplus \mathfrak{p}_{2\lambda}$. *Let* $X'_\mathfrak{m}$ *be the projection of an element* $X' \in \mathfrak{g}$ *on the space* $\mathfrak{m}$ *with respect to the expansion* $\mathfrak{g} = \mathfrak{k}_0 \oplus \mathfrak{m}$. *Expand the element* $[X, Y]_\mathfrak{m}$ *in the basis* (14). *Then its coordinates with respect to the elements X, Y are equal to zero.*

Proof. The relations (13) are evident. In view of the inclusions from Proposition 2 it is sufficient to prove the second statement only in the following cases: (a) $X = e_{\lambda,i}$, $Y = f_{2\lambda,j}$ and (b) $X = f_{\lambda,i}$, $Y = f_{2\lambda,j}$. Consider case (a). From (10) we get $[f_{2\lambda,j}, e_{\lambda,i}] \in \mathfrak{p}_\lambda$. The Ad_G-invariance of the metric $(\cdot,\cdot)$ gives $([f_{2\lambda,j}, e_{\lambda,i}], e_{\lambda,i}) = -(e_{\lambda,i}, [f_{2\lambda,j}, e_{\lambda,i}])$, $i = 1,\dots,q_1$, $j = 1,\dots,q_2$ and then $[f_{2\lambda,j}, e_{\lambda,i}] \perp e_{\lambda,i}$. Now, taking into account the orthogonality of the basis $\{e_{\lambda,i}\}_{i=1}^{q_1}$ of the space $\mathfrak{p}_\lambda$, we obtain the second statement in case (a). Case (b) is completely similar. □

5. Homogeneous submanifolds of two-body problem on two-point homogeneous compact Riemannian spaces

Let an operator H be as in Section 1; π_i, $i = 1, 2$ is the projection on the ith factor in the decomposition of $M = Q \times Q$, $\dim_{\mathbb{R}} Q = n$, and $\rho(x_1, x_2)$ the distance between the points $x_1, x_2 \in Q$. The function $\rho_2(x) := \rho(\pi_1(x), \pi_2(x))$, $x \in M$ determines the distance between particles. The free Hamiltonian H_0 for the system of two particles on the space M is the Laplace–Beltrami operator for the metric

$$g_2 := m_1\pi_1^* g + m_2\pi_2^* g$$

on this space, multiplied by $-1/2$, where $\pi_i^* g$ is the pullback of the metric g with respect to the projection on the ith factor. In order to find the explicitly invariant expression for the operator H_0, consider the foliation of the space M by submanifolds F_p that correspond to the constant level of the function ρ_2. The layer $F_p \subset M$ is G-homogeneous Riemannian manifolds with respect to the restriction of the metric g_2 on it; therefore, we can use the construction from Section 2. To "glue" these constructions for different p, we shall do the following. Choose a smooth curve $\gamma(p) \subset M$ to be transversal to the layers F_p, $p > 0$ and identify each F_p with the factor spaces G/K_p, where K_p is a stationary subgroup for the point γ_p. Let $\mathfrak{k}_p$ be a Lie algebra for K_p. Note that the layer F_0 is diffeomorphic to the space Q. Assume that the following condition is valid.

Condition 1. A function $p \to x_p$ on some interval $I \subset \mathbb{R}_+$ is a regular parameterization for the curve γ in M. This curve intersects each F_p, $p > 0$ once, and the set $M' := \cup_{p\in I} F_p$ is a connected dense open submanifold in M. Stationary subgroups for the points x_p coincide as $p \in I$.

For two-point homogeneous compact Riemannian spaces Q the curve from Condition 1 can be chosen in the following way. Let $\tilde{\gamma}(t) : (a, b) \subset \mathbb{R} \to Q$ be some geodesic on the space Q, where t is a natural parameter, $0 \in (a, b)$. The geodesic $\tilde{\gamma}$ realizes the strong minimum for length of curves between any two points on $\tilde{\gamma}$, if they are sufficiently close to each other [8]. Let $\tilde{\gamma}(t_1)$ and $\tilde{\gamma}(t_2)$ be two such points. Denote by Γ the segment of the geodesic γ between those points.

Let K be a subgroup of the group G consisting of all transformations conserving the points $\tilde{\gamma}(t_1)$ and $\tilde{\gamma}(t_2)$. Any isometry transforms a geodesic into a geodesic, so any element $q \in K$ conserves the segment Γ and consequently q conserves the whole geodesic $\tilde{\gamma}$. Thus, any point of Γ is a fixed point with respect to q, otherwise its motions along Γ would lead to changing the distance from this point to the ends of Γ. Consider the maximal interval $\Gamma' \supset \Gamma$, consisting of fixed points of K. The continuity of the K-action on the space Q implies that Γ' is closed. On the other hand, the group K conserves the geodesic $\tilde{\gamma}$, so it conserves those points on $\tilde{\gamma}$ near the ends of Γ' for which the distance from the ends of Γ' is realized only by $\tilde{\gamma}$. Thus the interval Γ' is open (as a subset $\tilde{\gamma}$) and coincides with $\tilde{\gamma}$, since the latter is connected.

Any geodesic is uniquely defined by any pair of its points, if they are close enough, therefore the group G acts transitively on the set of all geodesics of the space Q.

Let $s_1, s_2 : [0, \mathrm{diam}\, Q) \to (a, b)$ be smooth functions, s_1 is decreasing, s_2 is increasing, $s_1(0) = s_2(0) = 0$, and $\rho(\tilde{\gamma}(s_1(p)), \tilde{\gamma}(s_2(p))) \equiv p$, $p \in [0, \mathrm{diam}\, Q)$, $s_1'(p)^2 + s_2'(p)^2 \neq 0$. Define a curve $\gamma : [0, \mathrm{diam}\, Q) \to M$ by the formula $\gamma(p) = (\tilde{\gamma}(s_1(p)), \tilde{\gamma}(s_2(p))) \in M$. Below we shall formulate Condition 2, which implies that the stationary subgroup of the group G, corresponding to the points $\gamma(p)$, $p \in (0, \mathrm{diam}\, Q)$ and as shown above containing K, equals K. The validity of this condition will be verified later in Section 6. Obviously, the other requirements of Condition 1 are realized for $I = (0, \mathrm{diam}\, Q)$.

In this case we can identify the manifold M' with the space $I \times G/K$, where $K_{x_p} \equiv K$, $p \in I$, by the following formula:

$$I \times (G/K) \ni (p, bK) \leftrightarrow bx_p \in M'.$$

Let μ_2 be a measure on M, generated by the metric g_2. Using the identification above, carry this measure on the space $I \times (G/K)$, saving for it the same notation μ_2. Since the difference $M \setminus M'$ has a zero measure, we get the following isomorphism between spaces of measurable square integrable functions:

$$\mathcal{L}^2(M, \mu_2) \cong \mathcal{L}^2\left(I \times (G/K), \mu_2\right).$$

In the following, for simplicity it will be convenient to change the parameterization of the interval I using some function $p(r)$, $p'(r) \neq 0$, $r \in I' \subset \mathbb{R}_+$. In this case we will write $F_r := F_{p(r)}$. Since the group G acts only on the second factor of the space $M' = I \times (G/K)$, we can generalize the construction for the lift of differential operators from Section 2 and find for a G-invariant differential operator on the space $I \times (G/K)$ its lift onto the space $I \times G$.

Let $\mathfrak{p}$ be a subspace in $\mathfrak{g}$, complimentary to the subalgebra $\mathfrak{k} \equiv \mathfrak{k}_p$ such that $[\mathfrak{p}, \mathfrak{k}] \subset \mathfrak{p}$. Let $e_1, \ldots, e_{2n-1}$ be a basis in $\mathfrak{p}$, $X_1, \ldots, X_{2n-1}$ the corresponding Killing vector fields on the space M', and X_i^{l}, X_i^{r} the corresponding left- and right-invariant vector fields on the group G. Define a vector field tangent to the curve γ by the formula $X_0 = (\mathrm{d}/\mathrm{d}r)x_{p(r)}$. Since

$$\mathrm{d}L_q X_0 = \frac{\mathrm{d}}{\mathrm{d}r} L_q x_{p(r)} = \frac{\mathrm{d}}{\mathrm{d}r} x_{p(r)} = X_0, \quad \forall q \in K,$$

it is possible to spread the vector X_0 by left shifts to the whole space M' and obtain the smooth vector field on M' with the same notation X_0. The fields X_i, $i = 0, \ldots, 2n-1$ form the moving frame in some neighborhood of the curve $\gamma(p)$, $p \in (0, \mathrm{diam}\, Q)$, if the matrix Γ, consisting of the pairwise scalar products of the fields X_i, is nondegenerate on $\gamma(p)$, $p \in (0, \mathrm{diam}\, Q)$. Besides, at those points of the curve γ, where $\det \Gamma \neq 0$, the stationary subgroup of the group G, containing, as shown above, the group K, coincides with K, in view of the decomposition $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$. The next condition will be verified later in Section 6.

Condition 2. The matrix Γ is nonsingular on the curve $\gamma(p)$, $p \in (0, \mathrm{diam}\, Q)$.

Express the operator Δ_{g_2} via the moving frame $X_i, i = 0, \dots, 2n-1$ by the formula (7), assuming $\xi_i = X_i, i = 0, \dots, 2n-1$, and transform the result to the form $a^{ij} X_i \circ X_j + b^i X_i$. Since the field X_0, in contrast to other fields X_i, is not a Killing one, after calculations similar to (8), we obtain the following additional terms:

$$-(\pounds_{X_0} g_2)(X_i, X_j)\hat{g}_2^{0i} + \tfrac{1}{2}\hat{g}_2^{ki}(\pounds_{X_0} g_2)(X_k, X_i)\delta_j^0,$$

where $\hat{g}_{2,ij} := g_2(X_i, X_j)$, $0 \le i, j \le 2n-1$ are components of the metric g_2 with respect to the moving frame X_i. Taking into account $[X_0, X_i] = 0$, $\forall i = 0, \dots, 2n-1$, we get

$$(\pounds_{X_0} g_2)(X_i, X_j) = X_0 g_2(X_i, X_j) = X_0(\hat{g}_{2,ij}).$$

Thus, using formula (6), we obtain the following additional term in the formula for the operator:

$$\begin{aligned}
&\frac{1}{2} X_0(\hat{g}_{2,kj})\hat{g}_2^{kj}\hat{g}_2^{0i} X_i - X_0(\hat{g}_{2,kj})\hat{g}_2^{0k}\hat{g}_2^{ji} X_i \\
&\quad = \frac{1}{2\hat{\gamma}} X_0(\hat{\gamma})\hat{g}_2^{0i} X_i + X_0(\hat{g}^{0i}) X_i = \frac{1}{\sqrt{\hat{\gamma}}} X_0(\sqrt{\hat{\gamma}}\hat{g}_2^{0i}) X_i,
\end{aligned}$$

where $\hat{\gamma} = \det \hat{g}_{2,ij}$. Finally, we get

$$\Delta_{g_2} = \hat{g}^{ij} X_i \circ X_j + c_{jq}^q \hat{g}^{ji} X_i + \frac{1}{\sqrt{\hat{\gamma}}} X_0(\sqrt{\hat{\gamma}}\hat{g}_2^{0i}) X_i.$$

The field X_0 on the space $I' \times (G/K)$ has the form $\partial/\partial r$ and its lift on the space $I' \times G$ is tautological. This lift changes only the fields $X_i, i = 1, \dots, 2n-1$. According to Remark 2 and Lemma 1 we obtain the expression for the lift of the operator Δ_{g_2}

$$\tilde{\Delta}_{g_2} = \hat{g}^{ij}|_{x_0} X_i^{\mathrm{l}} \circ X_j^{\mathrm{l}} + (c_{jq}^q \hat{g}^{ji})|_{x_0} X_i^{\mathrm{l}} + \left.\left[\frac{1}{\sqrt{\hat{\gamma}}} X_0^{\mathrm{l}}(\sqrt{\hat{\gamma}}\hat{g}_2^{0i})\right]\right|_{x_0} X_i^{\mathrm{l}}, \tag{15}$$

where $X_0^{\mathrm{l}} := \partial/\partial r$.

The G-invariant measure μ_2 on the space $I' \times (G/K)$ has the form $\nu \otimes \mu$, where $\nu = \phi(r)\, \mathrm{d}r$ is the measure on the interval I', and μ is a G-invariant measure on the space G/K. The measure on the space $I' \times G$, corresponding to μ_2, has the form $\tilde{\mu}_2 = \nu \otimes \mu_G$, where μ_G is the left-invariant measure on the group G, appropriately normalized.

Similarly to Section 2, we can define the bijection ζ between the set of functions on the space $I' \times (G/K)$ and the set of functions on the space $I' \times G$ that are invariant with respect to the right K-shifts. Denote by $\mathcal{L}^2(I' \times G, K, \tilde{\mu}_2)$ the Hilbert space of square integrable K-invariant functions on $I' \times G$ with respect to the measure $\tilde{\mu}_2$ and the right K-shifts. Thus we obtain the following isometry of Hilbert spaces:

$$\zeta : \mathcal{L}^2(M, \mu_2) \to \mathcal{L}^2(I' \times G, K, \tilde{\mu}_2)$$

and also $\tilde{\Delta}_{g_2} \circ \zeta = \zeta \circ \Delta_{g_2}$.

6. Two-point Hamiltonian for the general compact two-point homogeneous space

In this section we shall obtain the concrete expression for the two-point Hamiltonian of the form (15) on the general compact two-point homogeneous space. Let

$$L, X_{\lambda,i}, Y_{\lambda,i}, X_{2\lambda,j}, Y_{2\lambda,j}, \quad i=1,\dots,q_1, \quad j=1,\dots,q_2 \tag{16}$$

be the Killing vector fields on the space Q, corresponding to the elements of the algebra $\mathfrak{g}$ from Proposition 4

$$\Lambda, e_{\lambda,i}, f_{\lambda,i}, e_{2\lambda,j}, f_{2\lambda,j}, \quad i=1,\dots,q_1, \quad j=1,\dots,q_2, \tag{17}$$

with respect to the left action of the group G on the space Q. Define the curve $\hat{\gamma}$ on the space Q by the formula $\hat{\gamma}(s)=\exp((s/R)\Lambda)x_0$. This curve will be the necessary geodesic $\tilde{\gamma}$ from the previous section according to the following proposition.

Proposition 5.

1. *Among all possible pairwise scalar products of fields* (16) *on the curve* $\hat{\gamma}$ *the nonzero products are the following*:

$$g(L,L)|_{\hat{\gamma}}=R^2, \tag{18}$$

$$g(X_{\lambda,i},X_{\lambda,i})|_{\hat{\gamma}}=\frac{R^2}{2}\left(1+\cos\frac{s}{R}\right), \quad i=1,\dots,q_1, \tag{19}$$

$$g(X_{\lambda,i},Y_{\lambda,i})|_{\hat{\gamma}}=-\frac{R^2}{2}\sin\frac{s}{R}, \quad i=1,\dots,q_1, \tag{20}$$

$$g(Y_{\lambda,i},Y_{\lambda,i})|_{\hat{\gamma}}=\frac{R^2}{2}\left(1-\cos\frac{s}{R}\right), \quad i=1,\dots,q_1, \tag{21}$$

$$g(X_{2\lambda,i},X_{2\lambda,i})|_{\hat{\gamma}}=\frac{R^2}{2}\left(1+\cos\frac{2s}{R}\right), \quad i=1,\dots,q_2, \tag{22}$$

$$g(X_{2\lambda,i},Y_{2\lambda,i})|_{\hat{\gamma}}=-\frac{R^2}{2}\sin\frac{2s}{R}, \quad i=1,\dots,q_2, \tag{23}$$

$$g(Y_{2\lambda,i},Y_{2\lambda,i})|_{\hat{\gamma}}=\frac{R^2}{2}\left(1-\cos\frac{2s}{R}\right), \quad i=1,\dots,q_2; \tag{24}$$

2. $\hat{\gamma}(s)=\tilde{\gamma}(s)$, $s\in[0,\operatorname{diam} Q)$.

Proof. By construction, L/R is the vector field tangent to the curve $\hat{\gamma}(s)$. Since

$$\frac{\mathrm{d}}{\mathrm{d}s}g(L,L)\bigg|_{\hat{\gamma}(s)}=\frac{2}{R}g([L,L],L)=0,$$

we have

$$g\left(\frac{1}{R}L,\frac{1}{R}L\right)\bigg|_{\hat{\gamma}(s)}\equiv g\left(\frac{1}{R}L,\frac{1}{R}L\right)\bigg|_{\hat{\gamma}(0)}=\left(\frac{1}{R}\Lambda,\frac{1}{R}\Lambda\right)=1$$

which is equivalent to (18), and so the parameter s is the natural parameter on the curve $\hat{\gamma}$. Using the equality

$$\frac{\mathrm{d}}{\mathrm{d}s}g(X,Y)\bigg|_{\hat{\gamma}(s)} = \frac{L}{R}(g(X,Y))\bigg|_{\hat{\gamma}(s)} = \frac{1}{R}(g([L,X],Y))\bigg|_{\hat{\gamma}(s)} + \frac{1}{R}(g(X,[L,Y]))\bigg|_{\hat{\gamma}(s)},$$

where X, Y are some vector fields on the curve $\hat{\gamma}$, the relations (13) and the connection of the metric $g(\cdot,\cdot)|_{T_{x_0}Q}$ with the scalar product $(\cdot,\cdot)$ on the algebra $\mathfrak{g}$, we obtain the system of linear differential equations with initial conditions with respect to all possible pairwise scalar products of the fields (16) on the curve $\hat{\gamma}$. This system decomposes to the set of easily solvable subsystems. For example, one obtains

$$\frac{\mathrm{d}}{\mathrm{d}s}g(X_{\lambda,i},X_{\lambda,i})\bigg|_{\hat{\gamma}(s)} = \frac{2}{R}g([L,X_{\lambda,i}],X_{\lambda,i})|_{\hat{\gamma}(s)} = \frac{1}{R}g(Y_{\lambda,i},X_{\lambda,i})|_{\hat{\gamma}(s)},$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}g(Y_{\lambda,i},X_{\lambda,i})\bigg|_{\hat{\gamma}(s)} &= \frac{1}{R}g([L,Y_{\lambda,i}],X_{\lambda,i})|_{\hat{\gamma}(s)} + \frac{1}{R}g(Y_{\lambda,i},[L,X_{\lambda,i}])|_{\hat{\gamma}(s)}\\ &= -\frac{1}{2R}g(X_{\lambda,i},X_{\lambda,i})|_{\hat{\gamma}(s)} + \frac{1}{2R}g(Y_{\lambda,i},Y_{\lambda,i})|_{\hat{\gamma}(s)},\\ \frac{\mathrm{d}}{\mathrm{d}s}g(Y_{\lambda,i},Y_{\lambda,i})\bigg|_{\hat{\gamma}(s)} &= \frac{2}{R}g([L,Y_{\lambda,i}],X_{\lambda,i})\bigg|_{\hat{\gamma}(s)} = -\frac{1}{R}g(X_{\lambda,i},Y_{\lambda,i})\bigg|_{\hat{\gamma}(s)}.\end{aligned}$$

Taking into account the initial conditions given by

$$\begin{aligned}&g(X_{\lambda,i},X_{\lambda,i})|_{\hat{\gamma}(0)} = (e_{\lambda,i},e_{\lambda,i}) = R^2,\\ &g(X_{\lambda,i},Y_{\lambda,i})|_{\hat{\gamma}(0)} = g(Y_{\lambda,i},Y_{\lambda,i})|_{\hat{\gamma}(0)} = 0, \quad i=1,\dots,q_1\end{aligned}$$

(valid due to the formula $Y_{\lambda,i}|_{\hat{\gamma}(0)} = 0$), we obtain (19)–(21). Other required formulae of the first statement can be obtained in a similar way.

Let us prove the equality $\hat{\gamma}(s) = \tilde{\gamma}(s)$. It is sufficient to show that $\nabla_L L|_{\hat{\gamma}(s)} = 0$, since the parameters of the curves $\hat{\gamma}(s)$, $\tilde{\gamma}(s)$ are natural. Formulae (9), (13) and the first statement of Proposition 5 imply

$$\begin{aligned}&g(\nabla_L L, X_{\lambda,i})|_{\hat{\gamma}(s)} = g(L,[L,X_{\lambda,i}])|_{\hat{\gamma}(s)} = \tfrac{1}{2}g(L,Y_{\lambda,i})|_{\hat{\gamma}(s)} = 0, \quad i=1,\dots,q_1,\\ &g(\nabla_L L, X_{2\lambda,j})|_{\hat{\gamma}(s)} = g(L,[L,X_{2\lambda,j}])|_{\hat{\gamma}(s)} = g(L,Y_{2\lambda,j})|_{\hat{\gamma}(s)} = 0, \quad j=1,\dots,q_2,\\ &g(\nabla_L L, L)|_{\hat{\gamma}(s)} = g(L,[L,L])|_{\hat{\gamma}(s)} = 0.\end{aligned} \tag{25}$$

Due to the first statement of this proposition the vector fields

$$L, X_{\lambda,i}, X_{2\lambda,j}, \quad i=1,\dots,q_1, \quad j=1,\dots,q_2$$

form a moving frame in the tangent spaces $T_{\hat{\gamma}(s)}Q$ as $s \in [0, \operatorname{diam} Q)$, since the matrix of their pairwise scalar products in these spaces is nonsingular. Thus, due to (25) we have: $\nabla_L L|_{\hat{\gamma}(s)} \equiv 0$, $s \in [0, \operatorname{diam} Q]$. □

Remark 5. It was mentioned above in Section 4 that the decomposition $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ is uniquely determined by the point x_0. Therefore, due to Proposition 5 and the isotropy of the space Q all nonzero elements of the space $\mathfrak{p}$ have the following property: the trajectories of all one-parameter subgroups corresponding to these elements and passing through the point x_0 are geodesics. In particular it holds for the elements $e_{\lambda,i}, e_{2\lambda,j}$, $i = 1, \dots, q_1$, $j = 1, \dots, q_2$.

Let us rename some notations to simplify the consideration of the space $M = Q \times Q$. Now, let

$$L = L^{(1)} + L^{(2)}, \qquad X_{\lambda,i} = X^{(1)}_{\lambda,i} \oplus X^{(2)}_{\lambda,i}, \qquad Y_{\lambda,i} = Y^{(1)}_{\lambda,i} \oplus Y^{(2)}_{\lambda,i}, \quad i = 1, \dots, q_1,$$
$$X_{2\lambda,j} = X^{(1)}_{2\lambda,j} \oplus X^{(2)}_{2\lambda,j}, \qquad Y_{2\lambda,j} = Y^{(1)}_{2\lambda,j} \oplus Y^{(2)}_{2\lambda,j}, \quad j = 1, \dots, q_2$$

be the decomposition of Killing vector fields on the space M, which correspond to the elements Λ, $e_{\lambda,i}$, $f_{\lambda,i}$, $e_{2\lambda,j}$, $f_{2\lambda,j}$ and the decomposition $TM = TQ \oplus TQ$. Let $\gamma(p)$ be a curve on the space M, constructed according to Section 5 with respect to the geodesic $\tilde{\gamma}$, and X_0 be the vector field on the space M constructed therein. Let $s_1(p) = \alpha p$, $s_2(p) = -\beta p$, $\alpha, \beta \in (0, 1)$, $\alpha + \beta = 1$, $p =: 2R \arctan r$, $r \in I'$, where $I' = (0, \infty)$ as $Q \neq \mathbf{P}^n(\mathbb{R})$ and $I' = (0, 1)$ as $Q = \mathbf{P}^n(\mathbb{R})$. Then

$$X_0 = \frac{\mathrm{d}}{\mathrm{d}r} x_{p(r)} = \frac{2}{1 + r^2}(\alpha L^{(1)} - \beta L^{(2)}), \tag{26}$$

since $\pi_k^* g(L, L) = R^2$, $k = 1, 2$ and $\tilde{\gamma}(s)$ is the normal parameterization of $\tilde{\gamma}$. Let us show that the vector fields

$$X_0, \quad L, \quad X_{\lambda,i}, \quad Y_{\lambda,i}, \quad X_{2\lambda,j}, \quad Y_{2\lambda,j}, \qquad i = 1, \dots, q_1, \quad j = 1, \dots, q_2 \tag{27}$$

form a moving frame in a neighborhood of the curve $\gamma(p)$, $p \in (0, \operatorname{diam} Q)$. To prove this, we shall find the matrix Γ of pairwise scalar products of these fields on the curve γ. Since $\pi_k^* g(L, L) = R^2$, $k = 1, 2$, one has

$$\begin{aligned}
g_2(X_0, X_0)|_\gamma &= g_2\left(\frac{2}{1 + r^2}(\alpha L^{(1)} - \beta L^{(2)}), \frac{2}{1 + r^2}(\alpha L^{(1)} - \beta L^{(2)})\right)\Bigg|_\gamma \\
&= \frac{4R^2}{(1 + r^2)^2}(\alpha^2 m_1 + \beta^2 m_2) =: a, \\
g_2(L, X_0)|_\gamma &= g_2\left(L^{(1)} + L^{(2)}, \frac{2}{1 + r^2}(\alpha L^{(1)} - \beta L^{(2)})\right)\Bigg|_\gamma \\
&= \frac{2R^2}{1 + r^2}(\alpha m_1 - \beta m_2) =: b, \qquad g_2(L, L)|_\gamma = (m_1 + m_2)R^2 =: c.
\end{aligned}$$

Due to (26) and the orthogonality of the fields $L^{(k)}$, $k = 1, 2$ with respect to all fields

$$X^{(k)}_{\lambda,i}, Y^{(k)}_{\lambda,i}, X^{(k)}_{2\lambda,j}, Y^{(k)}_{2\lambda,j}, \quad i = 1, \dots, q_1, \quad j = 1, \dots, q_2, k = 1, 2$$

we obtain the orthogonality of the fields X_0, L with respect to the fields

$$X_{\lambda,i}, \quad Y_{\lambda,i}, \quad X_{2\lambda,j}, \quad Y_{2\lambda,j}, \qquad i = 1, \dots, q_1, \quad j = 1, \dots, q_2. \tag{28}$$

Proposition 5 implies that among all possible pairwise scalar products of the fields (28) only products $(X_{\lambda,i}, Y_{\lambda,i})$, $i = 1, \dots, q_1$ and $(X_{2\lambda,j}, Y_{2\lambda,j})$, $j = 1, \dots, q_2$ can be nonzero. By simple calculations, taking into account (19)–(24), we obtain

$$\begin{aligned}
g_2(X_{\lambda,i}, X_{\lambda,i})|_\gamma &= R^2(m_1 \cos^2(\alpha \arctan r) + m_2 \cos^2(\beta \arctan r)) =: d,\\
g_2(X_{\lambda,i}, Y_{\lambda,i})|_\gamma &= R^2(-m_1 \sin(\alpha \arctan r) \cos(\alpha \arctan r)\\
&\quad + m_2 \sin(\beta \arctan r) \cos(\beta \arctan r)) =: h,\\
g_2(Y_{\lambda,i}, Y_{\lambda,i})|_\gamma &= R^2(m_1 \sin^2(\alpha \arctan r)\\
&\quad + m_2 \sin^2(\beta \arctan r)) =: f, \quad i = 1, \dots, q_1,\\
g_2(X_{2\lambda,j}, X_{2\lambda,j})|_\gamma &= R^2(m_1 \cos^2(2\alpha \arctan r) + m_2 \cos^2(2\beta \arctan r)) =: u,\\
g_2(X_{2\lambda,j}, Y_{2\lambda,j})|_\gamma &= R^2(-m_1 \sin(2\alpha \arctan r) \cos(2\alpha \arctan r)\\
&\quad + m_2 \sin(2\beta \arctan r) \cos(2\beta \arctan r)) =: w,\\
g_2(Y_{2\lambda,j}, Y_{2\lambda,j})|_\gamma &= R^2(m_1 \sin^2(2\alpha \arctan r) + m_2 \sin^2(2\beta \arctan r)) =: v,\\
&\quad j = 1, \dots, q_2.
\end{aligned}$$

Thus we conclude that the matrix $\Gamma = g_2|_\gamma$ has a block structure with the blocks:

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix} \text{ one time,} \qquad \begin{pmatrix} d & h \\ h & f \end{pmatrix} - q_1 \text{ times and } \begin{pmatrix} u & w \\ w & v \end{pmatrix} - q_2 \text{ times.}$$

We have therefore $\det \Gamma = (ac - b^2)(\mathrm{d}f - h^2)^{q_1}(uv - w^2)^{q_2}$. It is easy to show that

$$ac - b^2 = \frac{4R^4 m_1 m_2}{(1 + r^2)^2}, \qquad \mathrm{d}f - h^2 = \frac{R^4 m_1 m_2 r^2}{1 + r^2}, \qquad uv - w^2 = \frac{4R^4 m_1 m_2 r^2}{(1 + r^2)^2}.$$

Thus

$$\det \Gamma = \frac{4^{1+q_2}(R^4 m_1 m_2)^{1+q_1+q_2} r^{2(q_1+q_2)}}{(1 + r^2)^{2+q_1+2q_2}}, \tag{29}$$

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix}^{-1} = \frac{1}{4R^2 m_1 m_2} \begin{pmatrix} (1 + r^2)^2(m_1 + m_2) & 2(1 + r^2)(m_1\alpha - m_2\beta) \\ 2(1 + r^2)(m_1\alpha - m_2\beta) & 4(m_1\alpha^2 + m_2\beta^2) \end{pmatrix},$$

$$\begin{pmatrix} d & h \\ h & f \end{pmatrix}^{-1} = \begin{pmatrix} D_s & E_s \\ E_s & F_s \end{pmatrix}, \qquad \begin{pmatrix} u & w \\ w & v \end{pmatrix}^{-1} = \begin{pmatrix} C_s & B_s \\ B_s & A_s \end{pmatrix},$$

where

$$\begin{aligned}
D_s &= \frac{1 + r^2}{m_1 m_2 R^2 r^2}(m_1 \sin^2(\alpha \arctan(r)) + m_2 \sin^2(\beta \arctan(r))),\\
F_s &= \frac{1 + r^2}{m_1 m_2 R^2 r^2}(m_1 \cos^2(\alpha \arctan(r)) + m_2 \cos^2(\beta \arctan(r))),\\
E_s &= \frac{1 + r^2}{2m_1 m_2 R^2 r^2}(m_1 \sin(2\alpha \arctan(r)) - m_2 \sin(2\beta \arctan(r))),
\end{aligned}$$

$$
\begin{aligned}
C_s &= \frac{(1+r^2)^2}{4m_1m_2R^2r^2}(m_1\sin^2(2\alpha\arctan(r))+m_2\sin^2(2\beta\arctan(r))),\\
A_s &= \frac{(1+r^2)^2}{4m_1m_2R^2r^2}(m_1\cos^2(2\alpha\arctan(r))+m_2\cos^2(2\beta\arctan(r))),\\
B_s &= \frac{(1+r^2)^2}{8m_1m_2R^2r^2}(m_1\sin(4\alpha\arctan(r))-m_2\sin(4\beta\arctan(r))).
\end{aligned}
$$

In view of the formula (29), the fields (27) form a moving frame on the curve $\gamma(p)$, $p \in (0, \operatorname{diam} Q)$ and Condition 2 is satisfied. Let

$$
L^1,\, X^1_{\lambda,i},\, Y^1_{\lambda,i},\, X^1_{2\lambda,j},\, Y^1_{2\lambda,j}, \quad i=1,\dots,q_1, \quad j=1,\dots,q_2 \tag{30}
$$

be left-invariant vector fields on the group G, corresponding to elements (17) of the algebra $\mathfrak{g}$ and $X^1_0 = \partial/\partial r$ the vector field on I'. We consider the corresponding fields on the space $I' \times G$ saving the notations. The field X_0 commutes with all fields (27). So, due to Proposition 3, the expansion of a commutator $[X, Y]$, where X, Y are any elements of the frame (27), by the same frame, does not include X, Y. Thus, the second term in the lift of the two-body Hamiltonian on the space $I' \times G$ in accordance with (15) vanishes, since $c^q_{jq} = 0$ (even without summation over q). Consequently, this expression has the form:

$$
\begin{aligned}
\tilde{H}_0 = &-\frac{(1+r^2)^{1+(q_1/2)+q_2}}{8mR^2r^{q_1+q_2}}\frac{\partial}{\partial r}\circ\left(\frac{r^{q_1+q_2}}{(1+r^2)^{(q_1/2)+q_2-1}}\frac{\partial}{\partial r}\right)\\
&-\frac{(m_1\alpha-m_2\beta)(1+r^2)^{1+(q_1/2)+q_2}}{4m_1m_2R^2r^{q_1+q_2}}\left\{\frac{\partial}{\partial r}, \frac{r^{q_1+q_2}}{(1+r^2)^{(q_1/2)+q_2}}L^1\right\}\\
&-\frac{m_1\alpha^2+m_2\beta^2}{2m_1m_2R^2}(L^1)^2-\frac{1}{2}\sum_{i=1}^{q_1}(D_s(X^1_{\lambda,i})^2+F_s(Y^1_{\lambda,i})^2+E_s\{X^1_{\lambda,i}, Y^1_{\lambda,i}\})\\
&-\frac{1}{2}\sum_{j=1}^{q_2}(C_s(X^1_{2\lambda,j})^2+A_s(Y^1_{2\lambda,j})^2+B_s\{X^1_{2\lambda,j}, Y^1_{2\lambda,j}\}),
\end{aligned}\tag{31}
$$

where $\{X, Y\} = X\circ Y + Y\circ X$ is the anticommutator of X and Y, and $m := m_1/m_2$.

According to Section 5, the lift of the measure, generated by the metric g_2, on the space $I' \times G$ has the form $\tilde{\mu}_2 = \nu \otimes \mu_G$, where $\nu = \sqrt{\det \Gamma}\,\mathrm{d}r$ is the measure on I', and μ_G is the biinvariant measure on the group G. Changing, if necessary, the normalization we get $\nu = r^{q_1+q_2}\,\mathrm{d}r/(1+r^2)^{1+(q_1/2)+q_2}$. The calculations above can be summarized in the following theorem.

Theorem 4. *The quantum two-body Hamiltonian on a compact two-point homogeneous space Q with the isometry group G can be considered as the differential operator $\tilde{H}_0 + U(r)$ (where the operator $\tilde{H}_0$ on the space $I' \times G$ is given by the formula (31)), where $I' = (0, 1)$ in the case $Q = \mathbf{P}^n(\mathbb{R})$ and $I' = (0, \infty)$ in other cases, $\alpha, \beta \in (0, 1)$, $\alpha + \beta = 1$. Its domain of definition is dense in the space $\mathcal{L}^2(I' \times G, K, \tilde{\mu}_2)$, consisting of all square integrable K-invariant functions on $I' \times G$, with respect to the measure $\tilde{\mu}_2$ and the right K-shifts.*

7. Two-point Hamiltonian for the general noncompact two-point homogeneous space

Noncompact two-point homogeneous spaces of types 7–10 are analogous to the compact two-point homogeneous spaces of types 2–6, respectively. In particular, it means that Lie algebras $\mathfrak{g}$ of symmetry groups of analogous spaces are different real forms of a simple complex Lie algebra. The transition from one such real form to another can be done by multiplying the subspace $\mathfrak{p}$ from the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ by the imaginary unit $\mathbf{i}$ (or by $-\mathbf{i}$). In the space $M = Q \times Q$ this transition corresponds to the change $r \to \mathbf{i}r$, $R \to \mathbf{i}R$ [25]. For example, on $\mathbf{S}^2$ we have the transition from the metric

$$\frac{4R^2(\mathrm{d}r^2 + r^2\,\mathrm{d}\phi^2)}{(1+r^2)^2}, \quad r \in [0, \infty], \quad \phi \in \mathbb{R} \bmod 2\pi$$

to the metric

$$\frac{4R^2(\mathrm{d}r^2 - r^2\,\mathrm{d}\phi^2)}{(1-r^2)^2}, \quad r \in [0, 1), \quad \phi \in \mathbb{R} \bmod 2\pi$$

on the space $\mathbf{H}^2(\mathbb{R})$. It is clear that the analogous spaces have equal multiplicities q_1 and q_2. Thus, changing variables in (31) as

$$r \to \mathbf{i}r,\ R \to \mathbf{i}R,\ X^{\mathrm{l}}_{\lambda,i} \to -\mathbf{i}X^{\mathrm{l}}_{\lambda,i},\ X^{\mathrm{l}}_{2\lambda,j} \to -\mathbf{i}X^{\mathrm{l}}_{2\lambda,j},$$

we obtain

Theorem 5. *The quantum two-body Hamiltonian on a noncompact two-point homogeneous space Q with the isometry group G can be considered as the differential operator*

$$\begin{aligned}\tilde{H} = &-\frac{(1-r^2)^{1+(q_1/2)+q_2}}{8mR^2r^{q_1+q_2}}\frac{\partial}{\partial r} \circ \left(\frac{r^{q_1+q_2}}{(1-r^2)^{(q_1/2)+q_2-1}}\frac{\partial}{\partial r}\right) \\ &- \frac{(m_1\alpha - m_2\beta)(1-r^2)^{1+(q_1/2)+q_2}}{4m_1m_2R^2r^{q_1+q_2}}\left\{\frac{\partial}{\partial r}, \frac{r^{q_1+q_2}}{(1-r^2)^{(q_1/2)+q_2}}L^{\mathrm{l}}\right\} \\ &- \frac{m_1\alpha^2 + m_2\beta^2}{2m_1m_2R^2}(L^{\mathrm{l}})^2 - \frac{1}{2}\sum_{i=1}^{q_1}(D_h(X^{\mathrm{l}}_{\lambda,i})^2 + F_h(Y^{\mathrm{l}}_{\lambda,i})^2 + E_h\{X^{\mathrm{l}}_{\lambda,i}, Y^{\mathrm{l}}_{\lambda,i}\}) \\ &- \frac{1}{2}\sum_{j=1}^{q_2}(C_h(X^{\mathrm{l}}_{2\lambda,j})^2 + A_h(Y^{\mathrm{l}}_{2\lambda,j})^2 + B_h\{X^{\mathrm{l}}_{2\lambda,j}, Y^{\mathrm{l}}_{2\lambda,j}\}) + U(r), \end{aligned} \tag{32}$$

where

$$D_h = \frac{1-r^2}{m_1m_2R^2r^2}(m_1\sinh^2(\alpha\,\mathrm{arctanh}(r)) + m_2\sinh^2(\beta\,\mathrm{arctanh}(r))),$$

$$F_h = \frac{1-r^2}{m_1m_2R^2r^2}(m_1\cosh^2(\alpha\,\mathrm{arctanh}(r)) + m_2\cosh^2(\beta\,\mathrm{arctanh}(r))),$$

$$\begin{aligned}
E_h &= \frac{1-r^2}{2m_1m_2R^2r^2}(m_1\sinh(2\alpha\,\mathrm{arctanh}(r)) - m_2\sinh(2\beta\,\mathrm{arctanh}(r))),\\
C_h &= \frac{(1-r^2)^2}{4m_1m_2R^2r^2}(m_1\sinh^2(2\alpha\,\mathrm{arctanh}(r)) + m_2\sinh^2(2\beta\,\mathrm{arctanh}(r))),\\
A_h &= \frac{(1-r^2)^2}{4m_1m_2R^2r^2}(m_1\cosh^2(2\alpha\,\mathrm{arctanh}(r)) + m_2\cosh^2(2\beta\,\mathrm{arctanh}(r))),\\
B_h &= \frac{(1-r^2)^2}{8m_1m_2R^2r^2}(m_1\sinh(4\alpha\,\mathrm{arctanh}(r)) - m_2\sinh(4\beta\,\mathrm{arctanh}(r))),
\end{aligned} \tag{33}$$

acting on the space $I' \times G$, where $I' = (0, 1)$. Its domain of definition is dense in the space $\mathcal{L}^2(I' \times G, K, \tilde{\mu}_2)$, consisting of all square-integrable K-invariant functions on $I' \times G$, with respect to the measure $\tilde{\mu}_2 = \nu \otimes \mu_G$ and the right K-shifts. Now $\nu = r^{q_1+q_2}\,\mathrm{d}r/(1-r^2)^{1+(q_1/2)+q_2}$ is the measure on I' and μ_G is biinvariant measure on G, since G is unimodular.

The following remark is analogous to Remarks 4 and 5.

Remark 6. The space $\mathfrak{a} \oplus \mathfrak{p}_{2\lambda}$ generates in the space Q a completely geodesic submanifold of the constant sectional curvature $-R^{-2}$, isomorphic to the space $\mathbf{H}^{q_2+1}(\mathbb{R})$.

If $q_1 \neq 0$, the element Λ and an arbitrary nonzero element from the space $\mathfrak{p}_\lambda$ generate in Q a completely geodesic two-dimensional submanifolds of constant curvature $-R^{-2}$.

The trajectories of all one-parameter subgroups corresponding to the elements of the space $\mathfrak{p}$, passing through the point x_0, are geodesics. In particular, it holds for the elements $e_{\lambda,i}, e_{2\lambda,j}, i = 1, \dots, q_1, j = 1, \dots, q_2$.

8. The Hamiltonian function for the classical two-body problem on two-point homogeneous spaces

We can derive the Hamiltonian functions of classical two-body problems on two-point homogeneous spaces from (31) and (32). These functions are defined on the cotangent bundles $T^*(Q \times Q \setminus \mathrm{diag})$ and are polynomials of the second-order on each fiber.

The G-action on the space $(Q \times Q) \setminus \mathrm{diag}$ can be naturally lifted to the Poisson action on the space $T^*((Q\times Q)\setminus\mathrm{diag})$ [4,24]. It means that for any $X \in \mathfrak{g}$ there is a function p_X on the space $T^*((Q\times Q)\setminus\mathrm{diag}) \cong T^*I' \times T^*(G/K)$ linear on fibers. The Hamiltonian vector field, corresponding to this function, coincides with the lift of the Killing vector field for X onto the cotangent bundle. All such functions are integrals for all G-invariant Hamiltonian systems on $T^*I' \times T^*(G/K)$ and can be considered as the generalized momenta. The set of such functions is a Lie algebra with respect to the Poisson bracket, and the correspondence $X \to p_X$ is the isomorphism of Lie algebras. To obtain the classical Hamiltonian functions from quantum Hamiltonians we should change the left-invariant vector fields and the operator $\partial/\partial r$ to corresponding momenta, multiplied by the imaginary unit in formulae (31) or (32). Denote the momentum, corresponding to the operator $\partial/\partial r$, by p_r, and momenta corresponding to the fields (30), by

$$p_L,\ p_{x,\lambda,i},\ p_{y,\lambda,i},\ p_{x,2\lambda,j},\ p_{y,2\lambda,j}, \quad i=1,\dots,q_1, \quad j=1,\dots,q_2. \tag{34}$$

Then the Hamiltonian function for the classical two-body problem on two-point compact homogeneous spaces has the form:

$$\begin{aligned} H_s = {} & \frac{(1+r^2)^2}{8mR^2}p_r^2 + \frac{(m_1\alpha - m_2\beta)(1+r^2)}{4m_1m_2R^2}p_r p_L + \frac{m_1\alpha^2+m_2\beta^2}{2m_1m_2R^2}p_L^2 \\ & + \frac{1}{2}\sum_{i=1}^{q_1}(D_s(p_{x,\lambda,i})^2 + F_s(p_{y,\lambda,i})^2 + 2E_s p_{x,\lambda,i}p_{y,\lambda,i}) \\ & + \frac{1}{2}\sum_{j=1}^{q_2}(C_s(p_{x,2\lambda,j})^2 + A_s(p_{y,2\lambda,j})^2 + 2B_s p_{x,2\lambda,j}p_{y,2\lambda,j}) + U(r) \end{aligned} \tag{35}$$

and on the two-point noncompact homogeneous spaces other than the Euclidean one, it has the form:

$$\begin{aligned} H_h = {} & \frac{(1-r^2)^2}{8mR^2}p_r^2 + \frac{(m_1\alpha - m_2\beta)(1-r^2)}{4m_1m_2R^2}p_r p_L + \frac{m_1\alpha^2+m_2\beta^2}{2m_1m_2R^2}p_L^2 \\ & + \frac{1}{2}\sum_{i=1}^{q_1}(D_h(p_{x,\lambda,i})^2 + F_h(p_{y,\lambda,i})^2 + 2E_h p_{x,\lambda,i}p_{y,\lambda,i}) \\ & + \frac{1}{2}\sum_{j=1}^{q_2}(C_h(p_{x,2\lambda,j})^2 + A_h(p_{y,2\lambda,j})^2 + 2B_h p_{x,2\lambda,j}p_{y,2\lambda,j}) + U(r). \end{aligned} \tag{36}$$

This form of the Hamiltonian function is convenient for the Marsden–Weinstein reduction. It is clear that this reduction acts only on the second factor in the expansion of the phase space $T^*I' \times T^*(G/K)$. The description of reduced spaces for the space $T^*(G/K)$ with respect to this reduction was obtained in [14] in terms of the Ad_G^*-orbits. Take an arbitrary Ad_G^*-orbit $\mathcal{O}$ and find its submanifold $\mathcal{O}'$ annulled by the subalgebra $\mathfrak{k}$. The quotient space $\tilde{\mathcal{O}}$ of $\mathcal{O}'$ with respect to Ad_K^* action is isomorphic to the reduced phase space for the space $T^*(G/K)$. Hence reducing the Hamiltonian two-body system on two-point homogeneous spaces we obtain the Hamiltonian system on the space $T^*I' \times \tilde{\mathcal{O}}$.

Practically it means the following. Generalized momenta (34) corresponding to the elements of the basis in the space $\mathfrak{p} \subset \mathfrak{g}$ can be considered as linear functions on the annulator of the subalgebra $\mathfrak{k}$ in the space $\mathfrak{g}^*$ in view of the expansion $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ and the isomorphism $(\mathfrak{g}^*)^* \cong \mathfrak{g}$. Therefore the momenta (34) themselves can be considered as functions on the space $\tilde{\mathcal{O}}$. Combinations of these functions, independent on the space $\tilde{\mathcal{O}}$, are coordinates on $\tilde{\mathcal{O}}$ and their commutative relations define the symplectic structure on the space $\tilde{\mathcal{O}}$.

9. Mass center for two particles on two-point homogeneous spaces

The importance of the mass center concept for isolated system of particles or a rigid body in Euclidean space stems from the following properties:

1. it moves with a constant speed along a (geodesic) line for a classical mechanical system;
2. variables corresponding to the mass center are separated from other variables both in classical and quantum mechanical problems.

These properties imply, in particular, that the (generally complicated) motion of a system can be decomposed into the motion of one point representing the center of mass, and the motion of the system with respect to that point, often greatly simplifying the problem. Under the action of external forces the center of mass moves as if all forces act on the particle located at the center of mass and having the mass equal to the total mass of the system. An attempt to generalize the concept of the center of mass to the curved two-point homogeneous Riemannian spaces encounters difficulties related to the absence of nice dynamical properties such as 1 and 2 above. It is natural to define the mass center for the two particles on a two-point homogeneous Riemannian space as the point on the geodesic interval connecting these particles that divides the interval in definite ratio. If this ratio is equal to the ratio of particle masses, we denote the corresponding mass center by R_1.

However, even for spaces of constant sectional curvature, such a mass center does not have property 1 [25]. For example, consider two free particles on the sphere $\mathbf{S}^2$. Choose two antipodal points on the sphere (poles), and the equator connecting them. Let one point rest at the pole and another moves with the constant speed along the equator. Then any point on the interval connecting those particles does not move along geodesic unless this point coincides with one of the particles. The latter is obviously senseless. Therefore, for the mass center on a two-point homogeneous Riemannian space we must rely on properties different from the property 1.

9.1. Existing mass center concepts for spaces of a constant curvature

The axiomatic approach to the concept of mass center was developed in [26,27]. Let $\mathfrak{A} = \{(A_i, m_i)\}$ be a system (possibly empty) of material points A_i with masses m_i in the space Q of constant sectional curvature, which corresponds to the types 2 or 9 according to the classification given in Section 4. Denote by $\mathcal{A}$ the set of all such systems and by $\mathcal{A}_0$ the subset of one-particle systems. For any positive real number χ define the operation $\chi \cdot \mathfrak{A} = \{(A_i, \chi m_i)\}$.

Theorem 6 (Galperin [27]). *There is a unique map $\mathbb{U}$ of the set $\mathcal{A}$ onto the set $\mathcal{A}_0$, satisfying the following axioms*: (1) $\mathbb{U}(\{(A_1, m_1)\}) = \{(A_1, m_1)\}$, (2) $\mathbb{U}(\mathfrak{A} \cup \mathfrak{B}) = \mathbb{U}(\mathbb{U}(\mathfrak{A}) \cup \mathbb{U}(\mathfrak{B}))$, (3) $\mathbb{U}(\chi \cdot \mathfrak{A}) = \chi \cdot \mathbb{U}(\mathfrak{A})$, (4) $\mathbb{U} \circ q = q \circ \mathbb{U}$, (5) *the map $\mathbb{U}$ is continuous with respect to the natural topology on the space $\mathcal{A}$. Two systems are close to each other in this topology, if their material points are pairwise close and have similar masses. Points with small masses are close to the empty set.*

For the sphere $\mathbf{S}^n$ with unit curvature this map $\mathbb{U}$ takes the system $\{(A_1, m_1), (A_2, m_2)\}$ to the material point (*mass center*), *located on the geodesic interval connecting the points A_1, A_2 and dividing it in the ratio ρ_1/ρ_2, as measured from the point A_1. Besides,* $m_1 \sin(\rho_1) = m_2 \sin(\rho_2)$, $\rho_1 + \rho_2 = \rho$, *where ρ is the distance between particles and ρ_i, $i = 1, 2$ is the distance between ith particle and the mass center. The mass of the mass center is assumed to be* $\cos(\rho_1)m_1 + \cos(\rho_2)m_2$.

For the Lobachevski space $\mathbf{H}^n(\mathbb{R})$ *with unit curvature the map* $\mathbb{U}$ *is obtained by using the hyperbolic functions* $\sinh$, $\cosh$ *instead of the corresponding trigonometric functions* $\sin$, $\cos$.

This approach to the definition of the center of mass corresponds to the mass center concept in flat space–time of special relativity (SR) [27]. In fact, for a given inertial frame of reference, there exists a one to one correspondence between possible particle velocities in SR and material points in the space $\mathbf{H}^3(\mathbb{R})$, with masses equal to the rest masses in SR. Therefore, a system $\mathfrak{A} \in \mathcal{A}$ corresponds to a system $\varsigma(\mathfrak{A})$ of moving particles in SR. The total mass and momentum of the latter system uniquely determine the rest mass and velocity of some effective particle Ξ in SR. This particle determines the mass center $\varsigma^{-1}(\Xi)$ of the system $\mathfrak{A}$ in the space $\mathbf{H}^3(\mathbb{R})$. We denote the mass center defined in this way by R_2.

It is clear that this definition of a mass center can be easily generalized to systems with a distributed mass.

Note that the mass center R_2 of two particles with equal masses located at the diametrically opposite points of a sphere has an arbitrary position on the equator and the null mass, which is equivalent to the empty set.

The definition of the mass center R_2 seems to be quite natural. Unfortunately, no "good" dynamical properties are known for it. In order to find the mass center with such properties, we can try to search for a pure geometrical mass center without any mass. In this case we need not be concerned about the validity of axioms 2 and 4 of Theorem 6, and thus have more freedom. This approach to the mass center concept concerning the free motion on spaces $\mathbf{S}^n$, $\mathbf{H}^n(\mathbb{R})$, $n = 2, 3$ was developed to various degrees of generality in [28–30]. Consider the following definition of a mass center. Let $Q = \mathbf{H}^n(\mathbb{R})$, $n = 2, 3$. Define a rigid body in Q by a nonnegative density function $\varrho(x)$, $x \in Q$ with a compact connected support, and consider the function

$$\Upsilon(x) = \int_Q \sinh^2(\rho(x, y))\varrho(y)\,\mathrm{d}\mu, \tag{37}$$

where μ is the measure on the space Q, generated by the Riemannian metric. This function has a unique minimum and the coordinate of this minimum can be chosen as a definition of the center of mass R_3 for the rigid body. It is clear that the similar definition is also suitable for a system of particles.

Unlike the center of mass R_2, the mass center R_3 for two particles is determined from the equations $m_1 \sinh(2\rho_1) = m_2 \sinh(2\rho_2)$, $\rho_1 + \rho_2 = \rho$. Here as before ρ is the distance between the particles, and ρ_i, $i = 1, 2$ is the distance between the ith particle and the mass center located on the geodesic interval connecting the particles.

There are three types of actions of one parameter subgroups $\exp(tX)$, $X \in \mathfrak{g}$, $t \in \mathbb{R}$ of the group G in the hyperbolic space Q [31]. The one parameter subgroup, isomorphic to $\mathbb{S}^1$, conserves all points of a completely geodesic submanifold of codimension 2 and is called *rotation* around some geodesic (*an axis of a rotation*) for $Q = \mathbf{H}^3(\mathbb{R})$ or around some point (*a center of a rotation*) for $Q = \mathbf{H}^2(\mathbb{R})$. The corresponding element X is called *elliptic*. If a one-parameter subgroup, isomorphic to $\mathbb{R}$, conserves some geodesic then it is called a *transvection* along this geodesic (*an axis of a transvection*). The corresponding element X is called *hyperbolic*. The last type of action of a one-parameter subgroup is a parabolic action of $\mathbb{R}$. It shifts points of Q along the system of horocycles that are lines orthogonal at each point to all geodesics having a common point on the absolute. The corresponding element X is called *parabolic*.

Call a free movement of a rigid body a *free rotation* if all points of this body move along trajectories of some rotation. Call a free movement of a rigid body a *free transvection* if all points of this body move along trajectories of some transvection. The mass center R_3 has the following dynamical properties:

1. The free rotation of a rigid body around its mass center is possible in the space $\mathbf{H}^n(\mathbb{R})$. If $n = 2$, there is only one such rotation [29] and if $n = 3$ there are three different rotations [30] around three pairwise perpendicular axes passing through the mass center R_3.
2. All possible transvections of a rigid body have axes passing through the mass center R_3. For $n = 2$ there are two such geodesics. For $n = 3$ there are three such geodesics, and they coincide with the axes of free rotations.
3. The mass center R_3 is uniquely determined by any of the properties 1 or 2.
4. The velocities of all possible free rotations and transvections are constant.
5. There are no free movements of a rigid body along horocycles [29].

The situation for the spaces $Q = \mathbf{S}^n(\mathbb{R})$, $n = 2, 3$ is analogous if we restrict ourselves to rigid bodies of "moderate" sizes, i.e. if the diameter of a rigid body is no more than $\pi R/4$ [28]. This condition is required in order to differ transvections and rotations of rigid bodies by the location of immovable points of one parameter isometry subgroups with respect to the rigid body itself, since all such subgroups of the isometry group $\mathbf{SO}(n+1)$ are conjugated, and their trajectories in the space Q are equivalent.

Note that most free movements of a rigid body in spaces of constant sectional curvature do not correspond to the center of mass R_3 movement along a geodesic even when this rigid body is a homogeneous ball [28].

9.2. The connection of existing mass center concepts to the two-point Hamiltonian

Consider now the connection of formulae (31), (32) obtained for the two-body Hamiltonian to the mass center concepts. If we fix the parameter α, then the particle positions uniquely determine the location of the point $\tilde{\gamma}(0)$ in the space Q at every moment of time. This point divides the geodesic interval $\tilde{\gamma}(t)$, $t \in [s_1(s), s_2(s)]$ of length s in the ratio $\alpha/(1-\alpha)$. The left-invariant vector fields

$$L^1, X^1_{\lambda,i}, X^1_{2\lambda,j}, \quad i = 1, q_1, \quad j = 1, q_2 \tag{38}$$

on the group G in formulae (31), (32) correspond to the basis of the space $\mathfrak{p} \in \mathfrak{g}$. According to Remarks 5 and 6, the trajectories of one-parameter subgroups generated by those fields and passing through the point $\tilde{\gamma}(0)$ are geodesics. The dynamical approaches to the definition of a mass center considered above are based on a possible movement of a mass center along geodesics of the space Q. Therefore, dynamical properties of a point representing a potential candidate for the mass center role can be studied by identifying it with the point $\tilde{\gamma}(0)$. Such an identification can always be achieved by choosing the parameter α appropriately.

Definition 1. Let $Q_2 \subset Q \times Q$ be a set of two particle positions that correspond to the only one shortest path connecting particles. A map from Q_2 to Q is called the dynamical mass center if it maps a two particle position from Q_2 to the point on the geodesic interval connecting the particles that divides the length of this interval in some ratio depending only on particle masses. Besides, for any geodesic on Q and any interactive potential there should be some initial positions and velocities of particles such that this point moves along this geodesic with a constant speed. For brevity, we call the value of this map the "dynamical mass center".

Note that this definition is appropriate for any Riemannian space Q. For two-point homogeneous spaces the set Q_2 is open and dense in $Q \times Q$. According to what was stated above and in Section 8, the point $\tilde{\gamma}(0)$ moves along a geodesic with a constant speed if and only if the following equality holds:

$$\pi_2(\mathrm{d}H_{s,h}) = \omega\,\mathrm{d}p_L + \sum_{i=1}^{q_1} \omega'_i\,\mathrm{d}p_{x,\lambda,i} + \sum_{j=1}^{q_2} \omega''_j\,\mathrm{d}p_{x,\lambda,j}, \tag{39}$$

where $\pi_2(\mathrm{d}H_{s,h})$ is the projection of the differential $\mathrm{d}H_{s,h}$ of the function (35) or (36) onto the tangent space to the second factor of the expansion $T^*I' \times T^*(G/K)$, $\omega, \omega'_i, \omega''_j$ are some constants and $\omega^2 + \sum_{i=1}^{q_1}(\omega'_i)^2 + \sum_{j=1}^{q_2}(\omega''_j)^2 \neq 0$. It is clear that

$$\begin{aligned}\pi_2(\mathrm{d}H_{s,h}) = {} & \left(\frac{(m_1\alpha - m_2\beta)(1 \pm r^2)}{4m_1m_2R^2}p_r + \frac{m_1\alpha^2 + m_2\beta^2}{m_1m_2R^2}p_L\right)\mathrm{d}p_L \\ & + \sum_{i=1}^{q_1}[(D_{s,h}p_{x,\lambda,i} + E_{s,h}p_{y,\lambda,i})\,\mathrm{d}p_{x,\lambda,i} \\ & + (F_{s,h}p_{y,\lambda,i} + E_{s,h}p_{x,\lambda,i})\,\mathrm{d}p_{y,\lambda,i}] \\ & + \sum_{j=1}^{q_2}[(C_{s,h}p_{x,2\lambda,i} + B_{s,h}p_{y,2\lambda,i})\,\mathrm{d}p_{x,2\lambda,i} \\ & + (A_{s,h}p_{y,2\lambda,i} + B_{s,h}p_{x,2\lambda,i})\,\mathrm{d}p_{y,2\lambda,i}].\end{aligned} \tag{40}$$

In the case of an arbitrary potential $U(r)$ the variables r and p_r (and also the functions $A_{s,h}, B_{s,h}, C_{s,h}, D_{s,h}, E_{s,h}, F_{s,h}$) can take arbitrary values on a trajectory. Therefore, the equality (39) is possible only if $m_1\alpha - m_2\beta = 0$ and

$$\begin{aligned} & p_L = \text{const} \neq 0,\ p_{x,\lambda,i} = p_{y,\lambda,i} = 0,\ p_{x,2\lambda,j} = p_{y,2\lambda,j} = 0, \\ & \qquad i = 1, \dots, q_1, \quad j = 1, \dots, q_2. \end{aligned} \tag{41}$$

In view of commutative relations (13) the equalities (41) are conserved on a trajectory of the dynamical system. In this case $\mathrm{d}H_{s,h} \sim \mathrm{d}p_L$ and the motion of both particles is along the common geodesic.

The equality $m_1\alpha - m_2\beta = 0$ gives the ratio of the distances s_1 and s_2: $s_1/s_2 = m_2/m_1$, which corresponds to the mass center R_1. Thus only the mass center R_1 satisfies Definition 1. Note the connection of the mass center R_3 with the zeroes of coefficients $B_{s,h}$ and $E_{s,h}$.

If $B_s = 0$, we have $m_1 \sin(2s_1/R) = m_2 \sin(2s_2/R)$ which means the coincidence of the points $\tilde{\gamma}(0)$ and R_3. According to Remarks 4 and 6, the momenta $p_{x,2\lambda,j}$ and $p_{y,2\lambda,j}$ for some fixed j correspond to the instantaneous motion of particles along a two-dimensional completely geodesic submanifold of the constant curvature $\pm R^{-2}$.

If $E_s = 0$, we have $m_1 \sin(s_1/R) = m_2 \sin(s_2/R)$. Due to Remarks 4 and 6, momenta $p_{x,\lambda,i}$ and $p_{y,\lambda,i}$ for fixed i correspond to the instantaneous motion of particles along a two-dimensional completely geodesic submanifold of the constant curvature $\pm(2R)^{-2}$. Therefore, in this case also the point $\tilde{\gamma}(0)$ corresponds to the mass center R_3.

Let us note in conclusion that by appropriately choosing the parameter α, expressions (31) and (32) can be simplified such that coefficients $m_1\alpha - m_2\beta$, $B_{s,h}$ or $E_{s,h}$ vanish. These values of the parameter α correspond to the mass center concepts R_1 and R_3. In Euclidean case the choice $\alpha = m_2/(m_1 + m_2)$ leads to the separation of the variable r from other variables in two-point Hamiltonian. On two-point homogeneous spaces it is impossible to separate the variable r from other variables by means of the choice the parameter α.

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References

[1] S. Helgason, Differential operators on homogeneous spaces, Acta Math. 109 (1959) 239–299;
S. Helgason, Fundamental solutions of invariant differential operators on symmetric spaces, Am. J. Math. 86 (1964) 565–601.
[2] S. Helgason, Groups and Geometric Analysis, Academic Press, Orlando, FL 1984.
[3] S. Helgason, Geometric Analysis on Symmetric Spaces, AMS, Providence, 1994.
[4] E.B. Vinberg, Commutative homogeneous spaces and co-isotropic symplectic actions, Russ. Math. Surv. 56 (1) (2001) 1–60.
[5] G.E. Bredon, Introduction to Compact Transformation Groups, Academic Press, New York, 1972.
[6] M. Reed, B. Simon, Methods of modern mathematical physics, vol. I, Functional Analysis, Academic Press, New York, 1972.
[7] A.L. Besse, Einstein Manifolds, Springer, Berlin, 1987.
[8] S. Kobayashi, K. Nomizu, Foundations of Differential Geometry, vol. 1, Interscience, New York, 1963.
[9] V. Arnold, Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications a l'hydrodynamique des fluides parfaits, Annales de l'Institut Fourier 16 (1) (1966) 319–361.
[10] N.P. Landsman, Rieffel induction as generalized quantum Marsden–Weinstein reduction, J. Geom. Phys. 15 (1995) 285–319.
[11] K.K. Wren, Quantization of constrained systems with singularities using Rieffel induction, J. Geom. Phys. 24 (1998) 173–202.
[12] S. Tanimura, T. Iwai, Reduction of quantum systems on Riemannian manifolds with symmetry and application to molecular mechanics, J. Math. Phys. 41 (2000) 1814–1842.
[13] A.V. Shchepetilov, Quantum mechanical two body problem with central interaction on surfaces of constant curvature, Theor. Math. Phys. 118 (1999) 197–208.
[14] A.V. Shchepetilov, Two-body problem on spaces of constant curvature. I. Dependence of the Hamiltonian on the symmetry group and the reduction of the classical system, Theor. Math. Phys. 124 (2000) 1068–1981.
[15] I.E. Stepanova, A.V. Shchepetilov, Two-body problem on spaces of constant curvature: II. Spectral properties of the Hamiltonian, Theor. Math. Phys. 124 (2000) 1265–1272.
[16] S. Kobayashi, K. Nomizu, Foundations of Differential Geometry, vol. 2, Interscience, New York, 1969.

[17] A.A. Kirillov, Elements of the Theory of Representations, Springer, Berlin, 1975.
[18] J.A. Wolf, Spaces of Constant Curvature, University of California Press, Berkeley, CA, 1972.
[19] J. Tits, Sur certains classes d'espace homogenes de groupes de Lie, Acad. Roy. Belg. Cl. Sci. Mem. Coll. 29 (3) (1955).
[20] H.C. Wang, Two-point homogeneous spaces, Ann. Math. 55 (1952) 177–191.
[21] H. Matsumoto, Quelques remarques sur les espaces riemanniens isotropes, C. R. Acad. Sci. Paris. 272 (1971) 316–319.
[22] S. Helgason, Differential Geometry, Lie Groups, and Symmetric Spaces, Academic Press, New York, 1978.
[23] O.O. Loos, Symmetric Spaces, vol. II, Compact Spaces and Classification, W.A. Benjamen, Inc., New York, 1969.
[24] V. Arnold, Mathematical Methods of Classical Mechanics, Springer, Berlin, 1978.
[25] A.V. Shchepetilov, Reduction of the two-body problem with central interaction on simply connected spaces of constant sectional curvature, J. Phys. A 31 (1998) 6279–6291.
[26] G.A. Galperin, On the notion of centroid of material points system in spaces of constant curvature, Rep. Sov. Acad. Sci. 302 (1988) 1039–1044.
[27] G.A. Galperin, A concept of the mass center of a system of material points in the constant curvature spaces, Commun. Math. Phys. 154 (1993) 63–84.
[28] J. Zitterbarth, Some remarks on the motion of a rigid body in a space of constant curvature without external forces, Demonstratio Mathematica 24 (1991) 465–494.
[29] P. Nagy, Dynamical invariants of rigid motions on the hyperbolic plane, Geometriae Dedicata 37 (1991) 125–139.
[30] S. Salvai, On the dynamic of a rigid body in the hyperbolic space, J. Geom. Phys. 36 (2000) 126–139.
[31] N.L. Balazs, A. Voros, Chaos on the pseudosphere, Phys. Rep. 143 (1986) 109–240.